Data provided: formula sheet

MAS205

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2012–13**

Statistics Core **2 hours**

*Candidates should attempt **ALL** five questions.*
The maximum marks for the various parts of the questions are indicated.
The paper will be marked out of 100. (Q1–21; Q2–16; Q3–32; Q4–17; Q5–14)

1 Let U be a $Un(-a, a)$ random variable, for some constant $a > 0$.

(a) Find the distribution function of U. ***(4 marks)***

(b) The *kurtosis* of a random variable X with mean μ and variance σ^2 may be defined to be

$$\frac{E((X-\mu)^4)}{\sigma^4}.$$

Find the kurtosis of U. ***(6 marks)***

(c) Let $u_1, u_2, \ldots, u_n$ be a random sample from the distribution of U. Explain why the likelihood of a based on this random sample is

$$L(a; \mathbf{u}) = \begin{cases} \left(\dfrac{1}{2a}\right)^n & a \geq \max\limits_{1 \leq i \leq n} |u_i| \\ 0 & \text{otherwise,} \end{cases}$$

and hence give the maximum likelihood estimate of a. ***(11 marks)***

2 Let $X \sim Be(2, 2)$.

(a) Let $Y = 2X - 1$. Find the probability density function of Y. ***(8 marks)***

(b) Let $Z = Y^2$.

(i) For $0 < z < 1$, find $P(Z \leq z)$. ***(4 marks)***

(ii) Hence find the probability density function of Z. ***(4 marks)***

3 Let $T \subseteq \mathbb{R}^2$ be the set $\{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq \infty\}$, and let X and Y be random variables with joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} 6x^2(1-x)e^{-xy} & (x,y) \in T \\ 0 & \text{otherwise.} \end{cases}$$

(a) (i) Show that the marginal distribution of X is $Be(2,2)$, and that the conditional probability density function of Y given $X = x$ (for $0 < x < 1$) is $Ex(x)$. ***(10 marks)***

(ii) Hence use conditional expectation to find $E(Y)$. ***(7 marks)***

(b) Let $U = XY$ and $V = 2X$.

(i) Find the joint probability density function of the random vector $(U,V)^T$, stating clearly the values for which it is non-zero. ***(11 marks)***

(ii) Hence show that U and V are independent and give the distribution of U. ***(4 marks)***

4 Let the random vector $\mathbf{X} = (X,Y)^T$ have a bivariate normal distribution with mean vector $(2,2)^T$ and covariance matrix

$$\begin{pmatrix} 4 & -1 \\ -1 & 9 \end{pmatrix}.$$

(a) Give the means and variances of X and Y, and their correlation coefficient. ***(6 marks)***

(b) Let $W = X/2$ and $Z = X + 4Y$, and let $\mathbf{W} = (W,Z)^T$

(i) Find the covariance matrix of $\mathbf{W}$. ***(5 marks)***

(ii) Give the distribution of Z. ***(3 marks)***

(iii) Are W and Z independent? Give a reason for your answer. ***(3 marks)***

5 Let x_1 and x_2 be observations of independent Poisson random variables with parameters λ and 2λ respectively, where $\lambda > 0$ is unknown.

(a) Find the likelihood of λ given the data x_1, x_2. ***(3 marks)***

(b) Find the log likelihood of λ given the data x_1, x_2. ***(3 marks)***

(c) Find the maximum likelihood estimate of λ given the data x_1, x_2. ***(8 marks)***

End of Question Paper

SOME DISCRETE DISTRIBUTIONS

Name	**Genesis**	**Notation**	**Probability function**	$E(X)$	$\mathrm{Var}(X)$	**Applications**	**Comments**
Uniform (discrete)	Set of k equally likely outcomes (usually, not necessarily, the integers)	$U(1,...,k)$ (not standard)	$p(x) = 1/k$ $x = 1,...,k$	$\frac{k+1}{2}$	$\frac{k^2-1}{12}$	Dice	
Bernoulli trial	Expt. with two outcomes: 'success' w.p. θ and 'failure' w.p. $1-\theta$ $X \equiv$ no. successes	$Ber(\theta)$	$p(x) = \theta^x(1-\theta)^{1-x}$ $x = 0, 1$ $\theta\epsilon[0, 1]$	θ	$\theta(1-\theta)$	Coins, constituent of more complex distributions	
Binomial	$X \equiv$ no. successes in n ind. $Ber(\theta)$ trials	$Bi(n, \theta)$	$p(x) =^n C_x\theta^x(1-\theta)^{n-x}$ $x = 0, 1, 2, ..., n$ $\theta\epsilon[0, 1]$	$n\theta$	$n\theta(1-\theta)$	Sampling with replacement	$Bi(1, \theta) \equiv Ber(\theta)$
Geometric	$X \equiv$ no. failures until 1st success in sequence of ind. $Ber(\theta)$ trials	$\mathrm{Ge}(\theta)$	$p(x) = \theta(1-\theta)^x$ $x = 0, 1, 2, ...$ $\theta\epsilon[0, 1]$	$\frac{1-\theta}{\theta}$	$\frac{1-\theta}{\theta^2}$	Waiting times (for single events)	Alternative formulation in terms of $Y \equiv$ no. of trials to 1st success ($Y = X + 1$)
Negative binomial (or Pascal)	$X \equiv$ no. failures to mth success in sequence of ind. $Ber(\theta)$ trials. Generalization of Geometric	$Neg\ Bi\ (m, \theta)$ (not standard)	$p(x) =^{m+x-1} C_x\theta^m(1-\theta)^x$ $x = 0, 1, 2, ...$ $\theta\epsilon[0, 1]$	$\frac{m(1-\theta)}{\theta}$	$\frac{m(1-\theta)}{\theta^2}$	Waiting times (for compound events)	Neg $Bi(1, \theta) \equiv Ge(\theta)$ Remains valid for any $k > 0$ (not necessarily integer). Alternative formulation as above.
Hypergeometric	$X \equiv$ no. of defectives in sample of size n taken without replacement from population of size N of which N^* are defective	Hypergeom (N, N^*, n) (not standard, esp. order of arguments)	$p(x) = \frac{^{N^*}C_x\ ^{N-N^*}C_{n-x}}{^NC_n}$ $x = 0, ..., n$ (strictly $x = \max(0, n+N^*-N), ...,$ $..., \min(n, N^*)$)	$\frac{nN^*}{N}$	$\frac{N-n}{N-1} n \frac{N^*}{N}(1 - \frac{N^*}{N})$	Sampling without replacement	Sampling with replacement leads to the $Bi(n, \frac{N^*}{N})$ - a suitable approx if $\frac{n}{N} < 0.1$
Poisson	Arises empirically or via Poisson Process for counting events. For Po. Proc. rate ν the no. of events in time $t \sim Po(\nu t)$. Also as an approx. to the Binomial	$Po(\lambda)$	$p(x) = \frac{e^{-\lambda}\lambda^x}{x!}$ $x = 0, 1, 2, ...$ $\lambda > 0$	λ	λ	Counting events occurring 'at random' in space or time	$Bi(n, \theta) \equiv Po(n\theta)$ if n large, θ small

SOME CONTINUOUS DISTRIBUTIONS

Name	**Notation**	**p.d.f.**	$E(X)$	$\mathrm{Var}(X)$	**Applications**	**Comments**
Uniform (continuous) (or Rectangular)	$Un(\alpha, \beta)$	$f(x) = \dfrac{1}{\beta - \alpha}$ $x \in [\alpha, \beta]$ $\alpha < \beta$	$\dfrac{\alpha + \beta}{2}$	$\dfrac{(\beta - \alpha)^2}{12}$	Rounding errors $Un(-\frac{1}{2}, \frac{1}{2})$. Simulating other distributions from $Un(0,1)$.	
Exponential	$Ex(\lambda)$	$f(x) = \lambda e^{-\lambda x}$ $x > 0$ $\lambda > 0$	$\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$	Inter-event times for Poisson Process. Models lifetimes of non-ageing items.	Alternative parameterization in terms of $1/\lambda$ $Ga(1, \lambda) \equiv Ex(\lambda)$
Gamma	$Ga(\alpha, \beta)$	$f(x) = \dfrac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}$ $x \geq 0$ $\alpha, \beta > 0$	$\dfrac{\alpha}{\beta}$	$\dfrac{\alpha}{\beta^2}$	Times between k events for Poisson Process. Lifetimes of ageing items.	Alternative parameterization in terms of $1/\beta$ $Ga(1, \lambda) \equiv Ex(\lambda)$, $Ga(\nu/2, 1/2) \equiv X^2_\nu$,
Beta	$Be(\alpha, \beta)$	$f(x) = \dfrac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)}$ $x \in [0,1]$ $\alpha, \beta > 0$	$\dfrac{\alpha}{\alpha + \beta}$	$\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$	Useful model for variables with finite range. Also as a Bayesian conjugate prior.	$Be(1,1) \equiv Un(0,1)$ $Be(\alpha, \beta)$ is reflection about $\frac{1}{2}$ of $Be(\beta, \alpha)$. Can transform $Be(\alpha, \beta)$ on $[0,1]$ to any finite range $[a, b]$ by $Y = (b-a)X + a$
Normal	$N(\mu, \sigma^2)$	$f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\dfrac{x-\mu}{\sigma}\right)^2\right]$ $x \in (-\infty, \infty)$	μ	σ^2	Empirically and theoretically (via CLT etc.) a good model in many situations. Often easy to handle mathematically.	$X \sim N(\mu, \sigma^2) \Longrightarrow aX + b \sim N(a\mu + b, a^2\sigma^2)$ $\Longrightarrow Z = \frac{X-\mu}{\sigma} \sim N(0,1)$ So $P[X \in (u,v)] = P[Z \in \left(\frac{u-\mu}{\sigma}, \frac{v-\mu}{\sigma}\right)]$ $N(0,1)$ special case has p.d.f. denoted ϕ, c.d.f. Φ (tabulated). Note $\Phi(-z) = 1 - \Phi(z)$.
Chi-square	χ^2_ν	$f(x) = 2^{-\nu/2}\Gamma(\nu/2)^{-1}x^{\nu/2-1}e^{-x/2}$ $x > 0$ $\nu > 0$	ν	2ν	Sum of squares of ν standard normals	$X^2_\nu \equiv Ga(\nu/2, 1/2)$ If $X_1, X_2, \ldots, X_n \sim N(0,1)$ independent, then $\sum_{i=1}^n X_i^2 \sim X_n^2$
Student t	t_ν	$f(x) =$ $\nu^{-1/2} B\left(\frac{1}{2}, \frac{\nu}{2}\right)^{-1} \left(1 + x^2/\nu\right)^{-(\nu+1)/2}$ $x \in (-\infty, \infty)$ $\nu > 0$	0 (if $\nu > 1$)	$\dfrac{\nu}{\nu - 2}$ (if $\nu > 2$)	Useful alternative to Normal for variables with heavy tails.	If $X \sim N(0,1)$ and $Y \sim \chi^2_\nu$ independent then $\frac{X}{\sqrt{Y/\nu}} \sim t_\nu$. $t_1 \equiv$ Cauchy. $t^2_\nu \equiv F_{1,\nu}$.
F	$F_{\nu,\delta}$	$f(x) = \dfrac{\nu^{\nu/2}\delta^{\delta/2}x^{\nu/2-1}}{B(\nu/2, \delta/2)(\nu x + \delta)^{(\nu+\delta)/2}}$ $x > 0$ $\nu, \delta > 0$	$\dfrac{\delta}{\delta - 2}$ (if $\delta > 2$)	$\frac{2\delta^2(\nu+\delta-2)}{\nu(\delta-2)^2(\delta-4)}$ (if $\delta > 4$)	Scaled ratio of chi-squares. Used in tests to compare variances	If $X \sim \chi^2_\nu$ and $Y \sim \chi^2_\delta$ independent then $\frac{X/\nu}{Y/\delta} \sim F_{\nu,\delta}$. If $T \sim t_\nu$ then $T^2 \sim F_{1,\nu}$. If $Z \sim Be(\alpha, \beta)$ then $\frac{\beta Z}{\alpha(1-Z)} \sim F_{2\alpha, 2\beta}$.