The University Of Sheffield.

**SCHOOL OF MATHEMATICS AND STATISTICS** **Autumn Semester 2012–13**

**Advanced Calculus - MAS202** **2 hours**

*Attempt all the questions. The allocation of marks is shown in brackets.*

**1** (i) The random variables $X$ and $Y$ have joint density function

$$f(x, y) = \begin{cases} 2(2x+1)y^3 & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that

$$\int_0^1 \int_0^1 f(x, y)\, dxdy \;=\; 1\ .$$

(b) Calculate the probability $P\left(0 \le X \le \dfrac{1}{2}\right)$.

(c) Calculate the probability $P\,(X + Y \le 1)$.

***(14 marks)***

(ii) Let $\omega = (2xe^{x^2} + y^2)\, dx + 2xy\, dy$.

(a) Show, *without* finding a potential function, that $\omega$ is an exact differential.

(b) Now find a potential function $f$ for $\omega$.

(c) Evaluate the line integral $\displaystyle\int_\gamma \omega$ , where $\gamma : y = x^2$, $0 \le x \le 1$.

***(11 marks)***

**2** (i) State Green's Theorem, being careful to include any conditions needed for its validity. Hence evaluate

$$\int_C (x^2y + \cos(x)e^{\sin(x)+\sin(y)})dx + (x^3 + \cos(y)e^{\sin(x)+\sin(y)})dy,$$

where $C$ is the triangular path with vertices $(0,0)$, $(1,0)$ and $(0,1)$, described in the anticlockwise direction. ***(13 marks)***

(ii) Find and classify all the critical points of the function

$$f(x,y) = x^3 - 3x^2 - y^3 + 12y + 1.$$

***(12 marks)***

**3** (i) Let $f$ be the periodic function with period $2\pi$ such that $f(x) = x^2$ for $-\pi \leq x \leq \pi$.

(a) Sketch the graph of $f$. Calculate all the coefficients in the Fourier series for $f$. ***(15 marks)***

(b) Write the function $f$ as the Fourier series. What can you deduce by plugging in $x = 0$? ***(5 marks)***

(ii) A function $F$ is defined by the formula

$$F(x) = \int_1^{\sin(x^2)} e^{\cos(xt^2)}\, dt.$$

Write down an expression for the derivative $\frac{dF}{dx}$. (Do not attempt to evaluate the integral in your expression.) ***(5 marks)***

**4** (i) Find the Fourier transform $\mathcal{F}(f)(s) = \int_{-\infty}^{\infty} f(x)e^{-isx}dx$ of the function

$$f(x) = \begin{cases} 2\sin^2(\frac{x}{2}) & \text{if } 0 \leq x \leq 2\pi, \\ 0 & \text{otherwise.} \end{cases}$$

***(14 marks)***

(ii) Using $\mathcal{F}(f)(s)$ (or otherwise), find the integral $\int_{-\infty}^{\infty} \frac{e^{-2\pi is} - 1}{s(1-s^2)} e^{2is} ds$. ***(6 marks)***

(iii) Find the Taylor series of the function $f(x) = e^{2012x^{2013}}$ at $x = 0$. What is the coefficient of $x^{2012\times 2013}$ in the Taylor series of the function $f(x)$ at $x = 0$? ***(5 marks)***

## End of Question Paper