The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring semester 2012**

Stochastic Processes and Finance **3 hours**

Candidates may bring to the examination a calculator that conforms to University regulations.
Full marks may be obtained by complete answers to five questions. All answers will be marked, but credit will be given only for the best five answers. Total marks 150.

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1 (a) Let X be a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{G}$ and $\mathcal{H}$ be sub-σ-algebras of $\mathcal{F}$. The *conditional expectation* of X given $\mathcal{G}$ is defined to be the unique $\mathcal{G}$-measurable random variable $Y = \mathbb{E}(X|\mathcal{G})$ for which

$$\mathbb{E}(X 1_A) = \mathbb{E}(Y 1_A),$$

for all $A \in \mathcal{G}$.

(i) Explain what it means to say that $\mathcal{G}$ and $\mathcal{H}$ are independent. ***(3 marks)***

(ii) If $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$, show that

$$\mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{H}) = \mathbb{E}(X|\mathcal{H}).$$

(5 marks)

(iii) If $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$ and $\mathcal{G}$ and $\mathcal{H}$ are independent show that every set in $\mathcal{H}$ is either a null set or the complement of a null set. [Hint: use the definition of independence with both sets being the same.] ***(6 marks)***

(iv) Obtain $\mathbb{E}[(X - Z)^2|\mathcal{G}]$ as an explicit function of Z where Z is $\mathcal{G}$-measurable, X is independent of $\mathcal{G}$ and its distribution is given by $p_X(-1) = 1/4$, $p_X(1) = 1/4$ and $p_X(0) = 1/2$. ***(7 marks)***

(b) (i) Let $(Y_i, i \in \mathbb{N})$ be independent identically distributed random variables with finite mean $\mathbb{E}Y_i = \mu$ and let $\mathcal{F}_n = \sigma\{Y_1, Y_2 \ldots, Y_n\}$. Let

$$X_n = \sum_{i=1}^{n} Y_i.$$

When is $(X_n, n \in \mathbb{N})$ a submartingale with respect to $\mathcal{F}_n$? Justify your answer. ***(5 marks)***

(ii) Let Z_i be independent identically distributed positive random variables with mean one and with $\mathbb{E}|\log Z_i| < \infty$. Let

$$W_n = -\log(Z_1 \times Z_2 \times \ldots \times Z_n).$$

Show that W_n is a submartingale. (You may assume '$-\log$' is a convex function.) ***(4 marks)***

2 Throughout this question $(B(t), t \geq 0)$ is a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which is equipped with the natural filtration $(\mathcal{F}_t, t \geq 0)$ generated by the Brownian motion.

(a) What is meant here by 'natural filtration $(\mathcal{F}_t, t \geq 0)$'? ***(3 marks)***

(b) State precisely what is meant by describing $(B(t), t \geq 0)$ as a Brownian motion. ***(4 marks)***

(c) Let Y_1 be a $\mathcal{F}_1$-measurable random variable with $\mathbb{E}(Y_1^2) = c_1 < \infty$ and let Y_2 be a $\mathcal{F}_2$-measurable random variable with $\mathbb{E}(Y_2^2) = c_2 < \infty$. Define a stochastic process $(F(t), t \geq 0)$ as follows:

$$F(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 1 \\ Y_1 & \text{if } 1 < t \leq 2 \\ Y_2 & \text{if } 2 < t \leq 3 \\ 0 & \text{if } t > 3. \end{cases}$$

(i) Explain why $F(t)$ is an adapted process. ***(4 marks)***

(ii) Write

$$\int_0^3 F(s)dB(s),$$

as the sum of three random variables. ***(5 marks)***

(iii) Using the representation in the previous part obtain the mean and variance of

$$\int_0^3 F(s)dB(s),$$

justifying your calculations. ***(9 marks)***

(d) Find the stochastic differential of $B(t)e^{-B(t)^2}$. ***(5 marks)***

3 Throughout this question $(B(t), t \geq 0)$ is a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

(a) Consider the stochastic differential equation (SDE):

$$dY(t) = \frac{4}{1+t}dt + 2dB(t).$$

Write down the solution of this SDE with initial condition $Y(0) = 0$. Calculate the mean and variance of $Y(t)$ for $t > 0$. Hence, give the distribution of $Y(t)$.

(9 marks)

(b) For $0 \leq t \leq 1$, let

$$Y(t) = a(1-t) + bt + (1-t)\int_0^t \frac{1}{1-s}dB(s).$$

(i) Confirm that Y satisfies the SDE

$$dY(t) = \frac{b - Y(t)}{1-t}dt + dB(t).$$

(ii) Give the partial differential equation (PDE) associated with Y.

(iii) Describe the relationship between Y and the PDE.

(12 marks)

(c) Let $X(t)$ satisfy the SDE

$$dX(t) = \frac{X(t)}{2(1-t)}\left(1 - t - 2\log_e(X(t))\right)dt + X(t)dB(t).$$

If $Y(t) = \log_e(X(t))$ compute $dY(t)$ and hence obtain an SDE for Y. Use this and the result from part (b)(i) to solve the SDE for X when $X(0) = 1$.

(9 marks)

4 (a) The price of a stock at time n is denoted by $S(n)$. The *one step return* is defined by

$$K(n+1) = \frac{S(n+1) - S(n)}{S(n)}.$$

The price of a bond at time n is denoted by $B(n)$ with its *one step return* given by

$$R(n+1) = \frac{B(n+1) - B(n)}{B(n)}.$$

Define the discounted stock price at n to be $\widetilde{S}(n) = S(n)/B(n)$.

(i) Derive an expression for $S(n)$ in terms of the starting value $S(0)$ and the one-step returns $K(1), K(2), \ldots, K(n)$. ***(6 marks)***

(ii) Derive an expression for $S(n)$ in terms of $S(m)$, where $m < n$, and the one-step returns $K(m+1), \ldots, K(n)$. ***(2 marks)***

(iii) If the $K(i)$s are all independent random variables and are independent of $S(0)$, what can you say about the expected value of $S(n)$ when $S(m)$ is known? Justify your answer. ***(5 marks)***

(iv) Derive an expression for $\widetilde{S}(n)$ in terms of the starting value $\widetilde{S}(m)$ and the ratios $\tilde{k}(i) = (1 + K(i))/(1 + R(i)), \quad i = m+1, \ldots, n$. ***(4 marks)***

(v) If $\log \tilde{k}(i)$ are independent variables having the normal distribution with mean μ and variance σ^2 give the distribution of $\log \widetilde{S}(n)$ given the value of $\widetilde{S}(m)$. ***(4 marks)***

(b) In a finite market model there are $d+1$ financial assets $S_0, S_1, \ldots, S_d$. The asset prices are all adapted processes with respect to a given filtration $(\mathcal{F}_n, n = 0, 1, 2, \ldots, T)$.

(i) Give a precise mathematical definition of the *initial investment of a portfolio* and the *wealth process of a portfolio.* ***(3 marks)***

(ii) The wealth process of a self-financing portfolio ϕ is V_ϕ. Show that, with a suitable definition of $\phi(m) \cdot \Delta S(m)$, which you should give,

$$V_\phi(n) - V_\phi(0) = \sum_{m=1}^{n} \phi(m) \cdot \Delta S(m).$$

(6 marks)

5 Let $B = (B(t), 0 \leq T)$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which is equipped with the natural filtration $(\mathcal{F}_t, t \geq 0)$ generated by the Brownian motion. In the standard Black-Scholes model, the price $S(t)$ at time t of a stock is given by a geometric Brownian motion

$$S(t) = s \exp\{\sigma B(t) + \mu t\},$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$. We assume that there is a risk-free investment whose value at time t is $A(t) = e^{rt}$, where $r > 0$ is the interest rate.

(a) Find the stochastic differential equation (SDE) which is satisfied by the stock price. ***(5 marks)***

(b) Find the SDE which is satisfied by the discounted stock price $\widetilde{S}(t) = A(t)^{-1}S(t)$. ***(3 marks)***

(c) The original measure $\mathbb{P}$ is changed to the equivalent probability measure $\mathbb{Q}$ where

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left\{\int_0^T F(s)dB(s) - \frac{1}{2}\int_0^T F(s)^2 ds\right\},$$

and Girsanov's theorem then tells us that

$$W(t) = B(t) - \int_0^t F(s)ds$$

is a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{Q})$. Find $F = (F(t), 0 \leq t \leq T)$ for which $\widetilde{S}$ is a $\mathbb{Q}$-martingale. Justify your answer. ***(4 marks)***

(d) Show that

$$e^{-rT}S(T) = e^{-rt}S(t)\exp\left\{\sigma(W(T) - W(t)) - \frac{1}{2}\sigma^2(T-t)\right\}.$$

(5 marks)

(e) The arbitrage price at time t for a European contingent claim X (with $\mathbb{E}_{\mathbb{Q}}(X^2) < \infty$) is $e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}(X|\mathcal{F}_t)$. If $X = f(S(T))$, $S(t) = s$ and $\theta = T - t$, show that the arbitrage price of X at time t is given by

$$F(t, s) = e^{-r\theta}\mathbb{E}_{\mathbb{Q}}\left[f(se^{U+r\theta})\right]$$

with $U \sim N\left(-\frac{1}{2}\sigma^2\theta, \sigma^2\theta\right)$. ***(7 marks)***

(f) Give formulae in terms of F for the replicating portfolio (ϕ, ψ) for X at time t when $S(t) = s$. ***(6 marks)***

6 (a) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $(\mathcal{F}_n, n \in \mathcal{T})$, where $\mathcal{T} = \{0, 1, \ldots, T\}$. Let $Y = (Y(n), n \in \mathcal{T})$ be an adapted integrable process and define a new process Z which is called the *Snell envelope* of Y by

$$Z(T) = Y(T)$$
$$Z(n) = \max\{Y(n), \mathbb{E}(Z(n+1)|\mathcal{F}_n)\}, n = 0, 1, \ldots, T-1.$$

(i) Is Z a submartingale, supermartingale or a martingale? Justify your answer. ***(4 marks)***

(ii) The Snell envelope is applied to the pricing of American options. How are Y and Z interpreted in this case? ***(4 marks)***

(b) The term structure of a bond is the family of random variables $(P(t,T), 0 \leq t \leq T, T \geq 0)$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $P(t,T)$ is the value at t of a bond that pays one unit at T. How are the forward rate $f(t,T)$ and the spot-rate $r(t)$ obtained from $P(t,T)$? ***(4 marks)***

(c) In the same framework as in (b), define the discount factor by

$$A(t) = \exp\left(\int_0^t r(s)ds\right)$$

and the discounted term structure by $\widetilde{P}(t,T) = A(t)^{-1}P(t,T)$. Assume that there is a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ such that $(\widetilde{P}(t,T), 0 \leq t \leq T)$ is a $\mathbb{Q}$-martingale for a filtration $(\mathcal{F}_t, 0 \leq t \leq T)$ for each $T > 0$, and that the spot-rate is adapted to the filtration. Deduce that

$$P(t,T) = \mathbb{E}_{\mathbb{Q}}\left(\exp\left\{-\int_t^T r(s)ds\right\}\middle| \mathcal{F}_t\right).$$

(8 marks)

(d) Continuing the framework from (c), assume that

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left\{-\lambda B(t) - \frac{1}{2}\lambda^2 t\right\}$$

where $\lambda > 0$ and $(B(t), t \geq 0)$ is a Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Assume further that the spot-rate is given by the Vasicek model:

$$dr(t) = a(b - r(t))dt + \sigma dB(t),$$

where a, b and σ are positive constants. Let $b^* = b - (\lambda\sigma/a)$ and $X(t) = r(t) - b^*$.

(i) Describe the process $W(t) = B(t) + \lambda t$ on $(\Omega, \mathcal{F}, \mathbb{Q})$. ***(2 marks)***

(ii) Rewrite the Vasicek equation as a stochastic differential equation (SDE) driven by $W(t)$ instead of $B(t)$. Then make the substitution $X(t) = r(t) - b^*$ to obtain an SDE for $X(t)$. Solve this SDE for $X(t)$. ***(8 marks)***

End of Question Paper