The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2011–2012**

Fields - MAS438 **2 hours 30 minutes**

Answer ***four*** *questions. You are advised* ***not*** *to answer more than four questions: if you do, only your best four will be counted.*

1 (i) State the Subfield Criterion. ***(4 marks)***

(ii) For each of the subsets J_1, J_2 of $\mathbb{C}$ specified below determine, with justification, whether it is a subfield of $\mathbb{C}$:

(a) $J_1 = \{a + b\sqrt{2} : \ a, b \in \mathbb{Q}\}$, ***(5 marks)***

(b) $J_2 = \{a + bi\sqrt{2} + ci : \ a, b, c \in \mathbb{Q}\}$. ***(3 marks)***

(iii) Consider the subfield $L = \mathbb{Q}(\sqrt{2}, i\sqrt{2})$ of $\mathbb{C}$.

(a) Find $[L : \mathbb{Q}]$. Justify your answer and give a $\mathbb{Q}$-basis of L. ***(7 marks)***

(b) Prove that $L = \mathbb{Q}(i, \frac{1}{\sqrt{2}})$. ***(3 marks)***

(c) Find $\frac{1 + i\sqrt{2}}{1 - i\sqrt{2}}$. The answer should be given in terms of the basis of (a). ***(3 marks)***

2 (i) Let K be a subfield of a field L. Give a definition of $[L : K]$. ***(2 marks)***

(ii) State the degrees formula for finite field extensions $K \subseteq L \subseteq M$. ***(2 marks)***

(iii) Prove the degrees formula for finite field extensions $K \subseteq L \subseteq M$. ***(9 marks)***

(iv) Let $K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n$ be finite field extensions. Prove that

$$[K_n : K_0] = [K_n : K_{n-1}][K_{n-1} : K_{n-2}] \cdots [K_1 : K_0].$$ ***(5 marks)***

(v) Using (iv) (or otherwise) find $[\mathbb{Q}(\sqrt{2}, \sqrt{3}, i) : \mathbb{Q}]$. ***(7 marks)***

3 (i) (a) State Eisenstein's Irreducibility Criterion. ***(2 marks)***

(b) Prove Eisenstein's Irreducibility Criterion. ***(8 marks)***

(c) Show that the polynomial $x^5 - 15x^4 - 20x^3 - 35x^2 + 10$ is irreducible in $\mathbb{Q}[x]$. ***(2 marks)***

(ii) Give the definition of the minimal polynomial of an algebraic element a over a field K. ***(2 marks)***

(iii) Let $K \subseteq L$ be a field extension, and let $b \in L$ be algebraic over K. Let $m(x)$ be the minimal polynomial of b over K.

(a) Prove that $m(x)$ is irreducible in $K[x]$, and ***(6 marks)***

(b) if $p(x) \in K[x]$ has b as a root (that is, is such that $p(b) = 0$), then there exists $q(x) \in K[x]$ with $m(x)q(x) = p(x)$. ***(5 marks)***

4 (i) (a) Find the degree $[\mathbb{Q}(5^{\frac{1}{4}}) : \mathbb{Q}]$ and the minimal polynomial $m(x) \in \mathbb{Q}[x]$ of the element $5^{\frac{1}{4}}$ over $\mathbb{Q}$. ***(3 marks)***

(b) Find the degree $[\mathbb{Q}(5^{\frac{1}{4}}) : \mathbb{Q}(\sqrt{5})]$ and the minimal polynomial $m'(x) \in \mathbb{Q}(\sqrt{5})[x]$ of the element $5^{\frac{1}{4}}$ over $\mathbb{Q}(\sqrt{5})$. ***(6 marks)***

(c) Prove that $\mathbb{Q}(5^{\frac{1}{4}}) = \mathbb{Q}(a)$ where $a = 5^{\frac{1}{4}} + \sqrt{5}$. ***(5 marks)***

(ii) (a) Give the definition of the n-th cyclotomic field. Give a complex number which generates it. ***(3 marks)***

(b) Let p be a prime number. Prove that the p-th cyclotomic polynomial is irreducible in $\mathbb{Q}[X]$. ***(8 marks)***

5 (i) (a) Specify the four standard constructions that are used in the theory of ruler-and-compass constructions and involve perpendicular and parallel lines. ***(7 marks)***

(b) Give the definition of a constructible real number. ***(2 marks)***

(c) Let $a, b \in \mathbb{R}$ be constructible numbers. Using the four standard constructions prove that the numbers

$$a - b \text{ and } a + b$$

are constructible (You may use the fact that a point (x, y) is constructible if and only if the point (y, x) is constructible if and only if the numbers x and y are constructible). ***(6 marks)***

(d) Show that the number $1 + \sqrt{3 + \sqrt[4]{1 + \sqrt[4]{2}}}$ is constructible (You may use the fact that if x is constructible then so is $\sqrt{x}$). ***(4 marks)***

(ii) Let $a \in \mathbb{R}$. Prove that a is a constructible real number if and only if $(0, a)$ is a constructible point in the plane. ***(6 marks)***

End of Question Paper