The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS Spring Semester 2011-2012

Analytical Dynamics and Classical Field Theory 3 Hours

Answer five questions. If you answer more than five questions, only your best five will be counted.

1 (i) Write down the Euler-Lagrange equation and define all quantities which appear in this equation. ***(5 marks)***

(ii) Consider a particle of mass m, sliding on the inside of a bowl, which has the form of a parabolic surface of revolution with axis of symmetry vertical. Use the cylindrical polar coordinates (r, ϕ, z) to describe the particle. You are further given that the equation of the surface is $r^2 = az$, where a is a positive constant. First, find the kinetic energy of the particle. Secondly, given that the potential energy is mgz, where g is a positive constant, write down the Lagrange-function for the system. Finally, show that $mr^2\dot{\phi}$ is constant. ***(15 marks)***

2 (i) The Lagrange-function $L(q_i, \dot{q}_i, t)$ describes the dynamics of a system of N particles, with $i = 1...N$. It is a function of the generalised co-ordinates q_i, velocities $\dot{q}_i$ and time t. Define the canonical momenta P_i. Define the Hamilton-function H. ***(4 marks)***

(ii) Using the definition of the Hamilton-function and by explicit calculation show that the Hamilton-function is a function of q_i, P_i and time t only and derive the Hamilton-equations

$$\dot{q}_i = \frac{\partial H}{\partial P_i}, \qquad \dot{P}_i = -\frac{\partial H}{\partial q_i}, \qquad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \ .$$

(16 marks)

3 (i) Define the Poisson bracket $\{f, g\}$ between two functions $f(q_k, p_k)$ and $g(q_k, p_k)$, with $k = 1...N$, where N is the number of degrees of freedom. ***(2 marks)***

(ii) The angular momentum vector is given by $\vec{L} = \vec{r} \times \vec{p}$, where $\vec{r}$ is the position vector and $\vec{p}$ is the momentum vector. Evaluate $\{r_j, L_k\}$, where r_j is the j-component of $\vec{r}$ and L_k is the k-component of $\vec{L}$. Using your result, find $\{x, L_x\}$ and $\{x, L_y\}$. Hint: The k-component of the angular momentum can be written in the following form: $L_k = \sum_{n,m=1}^{3} \epsilon_{knm} r_n p_m$, where ϵ_{knm} is the antisymmetric symbol defined by $\epsilon_{123} = 1$ and $\epsilon_{knm} = 1$ if knm is a even permutation of 123, $\epsilon_{knm} = -1$ if knm is an odd permutation of 123 and 0 otherwise. ***(8 marks)***

(iii) You are given that f, g and h are functions of q and p. Verify the Jacobi identity for the Poisson brackets, i.e. show that

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$

(10 marks)

4 (i) A "Canonical Transformation" between the co-ordinates (q, p) and (Q, P) leaves what invariant? ***(3 marks)***

(ii) Consider a transformation of coordinates from $(q, \dot{q}, t)$ to $(Q, \dot{Q}, t)$.

(a) The Lagrange-functions in both co-coordinate systems are given by $L = L(q, \dot{q}, t)$ and $\mathcal{L} = \mathcal{L}(Q, \dot{Q}, t)$, respectively. Assume that Hamilton's principle is valid in both co-ordinate systems and therefore show that $L(q, \dot{q}, t) - \mathcal{L}(Q, \dot{Q}, t) = dF/dt$, where $F = F(q, Q, \dot{q}, \dot{Q}, t)$ is a general function. ***(3 marks)***

(b) You are given that the Hamilton-functions are given in both co-ordinate systems as $H = p\dot{q} - L$ and $\mathcal{H} = P\dot{Q} - \mathcal{L}$, with $p = \partial L/\partial \dot{q}$ and $P = \partial \mathcal{L}/\partial \dot{Q}$. Assume that the function F is a function of q, Q and time t only, i.e. $F = F(q, Q, t)$. Show that

$$p = \frac{\partial F}{\partial q}$$
$$P = -\frac{\partial F}{\partial Q}$$
$$\mathcal{H} = H + \frac{\partial F}{\partial t}$$

(10 marks)

(c) A relation between (q, p) and (Q, P) is given by $P = 1/q$ and $Q = pq^2$. Using the result from (b), find the generating function $F(q, Q, t)$. ***(4 marks)***

5 You are given that the Lorentz–transformation between two internal systems with co-ordinates x^μ (System A) and x'^μ (System B) is given by $\Lambda^\mu{}_\nu = \partial x'^\mu / \partial x^\nu$.

(i) How do a four-vector x^μ and a tensor $\sigma^{\mu\nu}$ transform under Lorentz-transformations? ***(5 marks)***

(ii) The four-velocity is defined as $u^\mu = dx^\mu/d\tau$ and the four-acceleration is defined as $a^\mu = du^\mu/d\tau$, where the proper time is defined as $c^2 d\tau^2 = \eta_{\mu\nu} dx^\mu dx^\nu$.

Show that the four-velocity is given by

$$u^\mu = \left(\frac{c}{\sqrt{1-(v/c)^2}}, \frac{\vec{v}}{\sqrt{1-(v/c)^2}} \right),$$

with $\vec{v} = d\vec{x}/dt$. Show that the four-acceleration can be written as

$$a^\mu = \left(\frac{(\vec{v}/c)\cdot(d\vec{v}/dt)}{(1-(v/c)^2)^2}, \frac{d\vec{v}/dt}{1-(v/c)^2} + \frac{(\vec{v}/c)((\vec{v}/c)\cdot(d\vec{v}/dt))}{(1-(v/c)^2)^2} \right)$$

(10 marks)

(iii) Use the expressions from 5 (ii) or otherwise to show that $u_\mu a^\mu = 0$. ***(5 marks)***

6 (i) Summarise the contents of Noether's theorem in words. ***(2 marks)***

(ii) A Lagrange-function $L(q_i, \dot{q}_i)$ describing N particles ($i = 1 \ldots N$) is not explicitly dependent on time t. Therefore, a transformation of the form $q_i \to q_i$ and $t \to t + \epsilon$ leaves the Lagrange-function invariant and therefore there exists a conserved quantity. Interpret the conserved quantity. You are given that the conserved quantity can in general be written as

$$\mathcal{Q} = \sum_{i=1}^{N} \frac{\partial L}{\partial \dot{q}_i} \Psi_i + \left(L - \sum_{i=1}^{N} \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) \Phi$$

for transformation of the form $q_i \to q_i + \epsilon \Psi_i(q_i, \dot{q}_i, t)$ and $t \to t + \epsilon \Phi(q_i, \dot{q}_i, t)$.

If the Lagrangian is invariant under the transformation $q_i \to q_i + \epsilon$ and $t \to t$, find the conserved quantity and its interpretation. ***(9 marks)***

(iii) Consider the following action, describing a scalar field:

$$S = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi) = \int d^4x \left(\frac{1}{2} \eta^{\mu\nu} (\partial_\mu \phi)(\partial_\nu \phi) - V(\phi) \right).$$

The potential energy $V(\phi)$ is given by $V(\phi) = \frac{1}{2} m^2 \phi^2$. Additionally, consider the following transformations with a constant parameter α of the form

$$\begin{aligned} x^\mu &\to e^\alpha x^\mu \\ \phi(x) &\to \phi(x) \exp(-\alpha). \end{aligned}$$

Show that the action above is invariant under these transformations only if $m = 0$. ***(9 marks)***

End of Question Paper