The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2011-2012**

Applicable Analysis **2 hours 30 minutes**

*Answer **four** questions. If you answer more than four questions, only your best four will be counted.*

You may use the following results when answering questions on this paper.

Table of Laplace Transforms	
Function	*Laplace Transform*
$t^\alpha e^{bt} (\alpha > -1)$	$\dfrac{\Gamma(\alpha+1)}{(s-b)^{\alpha+1}}$
$\sin at$	$\dfrac{a}{s^2+a^2}$
$\cos at$	$\dfrac{s}{s^2+a^2}$
$f(t)e^{bt}$	$F(s-b)$
$f^{(n)}(t)$	$s^n F(s) - \sum_{k=1}^{n} f^{(k-1)}(0) s^{n-k}$
$tf(t)$	$-F'(s)$

1 (i) Define what is meant by the statement that $\int_a^\infty f(x)\,dx$ exists.

(2 marks)

Prove, **from your definition,** each of the following statements:

$$\text{(a)} \quad \int_0^\infty \frac{1}{x^2+1}\,dx \quad \text{exists;}$$

$$\text{(b)} \quad \int_0^\infty \frac{x^3}{x^4+1}\,dx \quad \text{does not exist.}$$

(5 marks)

(ii) State, without proof, the Comparison Test for convergence and divergence of integrals of the form $\int_a^\infty f(x)\,dx$. Your statement should include conditions under which the results are valid.

(4 marks)

Prove each of the following, stating any standard results you need to use:

$$\text{(a)} \quad \int_0^\infty \frac{x^3(2-\sin x)}{x^4+1}\,dx \quad \text{diverges;}$$

$$\text{(b)} \quad \int_0^\infty \frac{1}{\sqrt{(1+x^4)}}\,dx \quad \text{converges.}$$

(7 marks)

(iii) Prove each of the following, stating any standard results you need to use:

$$\text{(a)} \quad \int_0^1 \ln x\,dx \quad \text{converges;}$$

$$\text{(b)} \quad \int_0^1 \frac{e^{-x}}{x\sqrt{x}}\,dx \quad \text{diverges.}$$

(7 marks)

2 (i) State, without proof, the theorem concerning change of order in a repeated integral of the form

$$\int_c^d dy \int_a^\infty f(x,y)\,dx\,.$$

Your statement should include conditions under which the result holds. ***(2 marks)***

Let $0 < c < d$. Prove that

$$\int_c^d dy \int_0^\infty \frac{y}{x^2+y^2}\,dx \quad = \quad \int_0^\infty dx \int_c^d \frac{y}{x^2+y^2}\,dy\,.$$ ***(5 marks)***

Hence, or otherwise, show that

$$\int_0^\infty \ln\left(\frac{x^2+d^2}{x^2+c^2}\right)\,dx \quad = \quad \pi(d-c)\,.$$ ***(6 marks)***

(ii) Define the Γ function. ***(2 marks)***

Prove that

(a) $\displaystyle\int_0^\infty xe^{-x^6}\,dx \quad = \quad \tfrac{1}{6}\,\Gamma(\tfrac{1}{3})\ ;$

(b) $\displaystyle\int_0^1 \frac{x}{\sqrt{\ln\left(\frac{1}{x}\right)}}\,dx \quad = \quad \sqrt{\frac{\pi}{2}}\,.$

(10 marks)

3 (i) Define the Beta function. State, without proof, the relation between the Beta and Gamma functions. ***(3 marks)***

Prove that

$$B(x,y) = 2\int_0^{\pi/2} \cos^{2x-1}\theta \sin^{2y-1}\theta \, d\theta \qquad (x>0,\ y>0)$$

and

$$B(x,y) = \int_0^{\infty} \frac{u^{x-1}}{(1+u)^{x+y}} \, du \qquad (x>0,\ y>0).$$

(4 marks)

Prove each of the following, stating any standard results you need to use:

(a) $$\int_0^{\pi/2} \frac{1}{\sqrt{\cos\theta \, \sin\theta}} \, d\theta = \frac{\left[\Gamma(\frac{1}{4})\right]^2}{2\sqrt{\pi}};$$

(b) $$\int_0^{\infty} \frac{t^4}{(1+t^4)^2} \, dt = \frac{\pi\sqrt{2}}{16}.$$

(9 marks)

(ii) Define what is meant by the statement that $\int_0^{\infty} e^{-st} f(t)\, dt$ has abscissa of convergence c. ***(2 marks)***

Prove that $\int_0^{\infty} e^{-st} \frac{t^2}{t^3+1} \, dt$ has abscissa of convergence 0. ***(7 marks)***

4 (i) In each of the following cases, find the function continuous on $[0,\infty)$, with the given Laplace transform:

(a) $\dfrac{2}{s^2-1} \quad (s>1)\,;$

(b) $\dfrac{s}{s^2+2s+10} \quad (s>1)\,;$

(c) $\dfrac{2s}{(s^2+4)^2} \quad (s>0)\,.$

(8 marks)

(ii) Let $f:[0,\infty)\to\mathbb{R}$ be continuous and suppose that the Laplace transform $F=L(f)$ exists on (c,∞) for some $c\in\mathbb{R}$. State, without proof, the formula giving $L\left(\dfrac{f(t)}{t}\right)$ in terms of F. Your statement should include sufficient conditions to ensure the validity of the formula. ***(2 marks)***

Show that

$$L\left(\frac{1-e^t}{t}\right) \;=\; \ln\left(\frac{s-1}{s}\right) \quad (s>1)\,.$$ ***(6 marks)***

(iii) Let $b>0$. Verify that

$$\int_0^\infty \frac{\sqrt{x}}{x^2+b^2}\,dx \;=\; \frac{\pi\sqrt{2}}{2\sqrt{b}}\,.$$ ***(3 marks)***

By considering $\displaystyle\int_0^\infty \frac{\sin(xt)}{\sqrt{x}}\,dx$, show that

$$\int_0^\infty \frac{\sin x}{\sqrt{x}}\,dx \;=\; \sqrt{\frac{\pi}{2}}\,.$$ ***(6 marks)***

5 (i) Suppose the functions f and g are continuous on $[0, \infty)$. Define the convolution $f * g$.

State, without proof, a relation between $L(f * g)$, $L(f)$ and $L(g)$. ***(3 marks)***

Find the function f continuous on $[0, \infty)$ such that

$$\int_0^t f(u)\,(t-u)\,du \quad = \quad t \sin t \quad (t \geq 0)\,.$$ ***(9 marks)***

(ii) Using Laplace transforms, solve the differential equation

$$t\,\frac{d^2y}{dt^2} \;+\; t\,\frac{dy}{dt} \;+\; y \;=\; e^{-t}$$

subject to the initial conditions $y(0) = 1$ and $y(1) = \dfrac{2}{e}$. ***(13 marks)***

End of Question Paper