The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS Spring Semester 2011–2012

Fluid Mechanics I 2 hours

*Marks will be awarded for your best **four** answers.*

1 (i) Consider a steady flow of the following form

$$\boldsymbol{u} = (4x, 2y, z)\,.$$

(a) Solve the following set of equations for $\boldsymbol{x}$ to find the particle paths subject to the flow $\boldsymbol{u}$

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{u}(\boldsymbol{x}),$$

where

$$\boldsymbol{x} = (a, b, c)$$

at $t = 0$. ***(5 marks)***

(b) Show that a unit sphere will evolve into a spheroid and determine its volume $V(t)$ as a function of t. ***(5 marks)***

(c) Confirm that

$$\frac{1}{V(t)}\frac{dV}{dt} = \text{div }\boldsymbol{u}.$$

(5 marks)

(ii) We consider a stress tensor

$$\sigma_{ij} = C_{ij} + C_{ijkl}e_{kl}$$

of a Newtonian fluid, where C_{ij}, C_{ijkl} are constants and e_{kl} is the strain rate tensor. We note that when the fluid is at rest we have

$$\sigma_{ij} = -p\delta_{ij},$$

where p denotes the pressure and δ_{ij} is Kronecker's delta.

(a) Assuming isotropy of the fluid we can write

$$C_{ij} = A\delta_{ij},$$

$$C_{ijkl} = B\delta_{ij}\delta_{kl} + C\delta_{ik}\delta_{jl} + D\delta_{il}\delta_{jk},$$

where A, B, C, D are constants. Derive the following expression

$$\sigma_{ij} = (-p + \lambda e_{kk})\delta_{ij} + 2\mu e_{ij}$$

by suitably defining two constants λ, μ. ***(6 marks)***

(b) Take the trace of σ_{ij}, and, by defining $\zeta = \lambda + \frac{2}{3}\mu$, eliminate λ from the result. Give the physical meaning of ζ. ***(4 marks)***

2 Consider the three-dimensional Euler equations and the continuity equation for an incompressible velocity field $\boldsymbol{u}$,

$$\frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{u} \times \boldsymbol{\omega} - \nabla\left(p + \frac{|\boldsymbol{u}|^2}{2}\right),$$

$$\nabla \cdot \boldsymbol{u} = 0,$$

where p denotes the pressure and $\boldsymbol{\omega}$ the vorticity.

(i) Derive the vorticity equations.

If necessary, you may use the following vector identity:

$$\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = -(\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} + \boldsymbol{A}(\nabla \cdot \boldsymbol{B}) - \boldsymbol{B}(\nabla \cdot \boldsymbol{A}).$$

(6 marks)

(ii) In the two-dimensional case

$$\boldsymbol{u} = (u(x, y, t), v(x, y, t), 0),$$

$$\boldsymbol{\omega} = (0, 0, \omega(x, y, t)),$$

where

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

is the scalar vorticity, show that the vorticity equation can be written as

$$\frac{\partial \omega}{\partial t} + J(\omega, \psi) = 0.$$

Here, ψ is the stream function $\boldsymbol{u} = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right)$ and

$$J(\omega, \psi) = \frac{\partial \omega}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial \omega}{\partial y}\frac{\partial \psi}{\partial x}$$

denotes a Jacobian determinant. ***(6 marks)***

(iii) In the special case where the vorticity is a function of the stream function $\omega = f(\psi)$ with an arbitrary smooth function f, show that ω is a steady solution.

(4 marks)

(iv) Under the same condition, show that

$$\int^{\psi} \omega(\psi')\,d\psi' = -\left(p + \frac{|\boldsymbol{u}|^2}{2}\right) + C,$$

where C is a constant. ***(9 marks)***

3 We consider a steady flow between two parallel plates $y = 0, h$, which is governed by the incompressible Navier-Stokes equations

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \boldsymbol{u},$$

$$\nabla \cdot \boldsymbol{u} = 0.$$

Here ρ denotes the uniform density of fluid and $\nu (= \mu/\rho)$ kinematic viscosity. The flow is driven by a constant pressure gradient $\alpha = -\dfrac{\partial p}{\partial x} (> 0)$.

(i) Assuming that the velocity field has the form $\boldsymbol{u} = (u(x, y), 0, 0)$, write down each component of the Navier-Stokes equations and the continuity equation. ***(5 marks)***

(ii) Show that the velocity has the form $u = u(y)$ and the pressure $p = p(x)$. By taking into account the boundary conditions $u = 0$ at $y = 0, h$, solve the above equations for the velocity and the pressure. ***(9 marks)***

(iii) Compute the net volume flux, that is, the volume of fluid passing between the plates in the x-direction per unit time, per unit length in the z-direction. ***(3 marks)***

(iv) Compute the viscous shear stress at $y = 0$. ***(4 marks)***

(v) Assume we place material line elements aligned in the x- and in the y-directions. Determine whether they are instantaneously stretched or not by checking the strain rate tensor. ***(4 marks)***

4 You are given the two-dimensional Navier-Stokes equations and the continuity equation in terms of cylindrical polar co-ordinates (r, θ) with standard notations :

$$\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right),$$

$$\frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left(\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right),$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0,$$

where

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Consider a steady flow in the region between two infinite coaxial circular cylinders of radii a and b (with $a < b$), where the inner cylinder is rotating with constant angular velocity Ω and the outer cylinder is at rest. We assume that the velocity field is given by

$$\boldsymbol{u} = u(r)\boldsymbol{e}_\theta.$$

(i) Confirm that the continuity equation is satisfied. ***(2 marks)***

(ii) Simplify each component of the Navier-Stokes equations, assuming that $\dfrac{\partial p}{\partial \theta} = 0$. ***(6 marks)***

(iii) Show that the equation for $u(r)$ can be written as

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(ru)\right) = 0. \qquad (1)$$

(6 marks)

(iv) By solving the above equation (1) show that the velocity has the form

$$u(r) = \frac{a^2\Omega}{b^2 - a^2}\left(\frac{b^2}{r} - r\right), \quad (a \le r \le b).$$

(6 marks)

(v) Using that fact that the relevant component of the stress tensor is given by

$$\sigma_{r\theta} = \mu\left(\frac{du}{dr} - \frac{u}{r}\right),$$

where μ is the viscosity, find the torque (per unit length of the cylinder) required to maintain the rotational motion. ***(5 marks)***

5 (i) A fluid moves in a steady two-dimensional flow in the region defined by $x \geq 0$, $y \geq 0$. The boundary with equation $y = 0$ is occupied by a stationary flat plate. Given that the x-component of velocity $u \to U$ as $y \to \infty$, where U is a constant, write down the expressions for

(a) the displacement thickness δ_1 of the boundary layer,

and

(b) the momentum thickness δ_2 of the boundary layer.

(4 marks)

(c) When the x-component of the flow is given by

$$\frac{u}{U} = \begin{cases} \left(\frac{y}{\delta}\right)^{1/2}, & \text{for } 0 \leq y \leq \delta \\ 1, & \text{for } y \geq \delta \end{cases}$$

where δ is the boundary layer thickness, compute δ_1 and δ_2 explicitly.

(6 marks)

(ii) Consider a steady two-dimensional flow past a semi-infinite solid boundary along $y = 0$ in the region $y \geq 0$, $x \geq 0$. Blasius's boundary layer has the following properties:

$$u \to U \text{ as } y \to \infty, \text{ for all } x \geq 0,$$

$$u = v = 0 \text{ at } y = 0, \text{ for all } x \geq 0,$$

$$u = U \text{ at } x = 0, \text{ for all } y \geq 0,$$

where u, v are the x- and y- components of the velocity and U is a constant. The equations for the velocity are given by

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2},$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

(a) Assuming $v \to v_\infty$ as $y \to \infty$, derive the expression for v_∞ in terms of $\frac{\partial u}{\partial x}$.

(5 marks)

(b) Derive the momentum equation for the boundary layer

$$\frac{d}{dx}\int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy = \frac{\tau_w}{\rho U^2},$$

where

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0}$$

is the skin friction.

(10 marks)

End of Question Paper