The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2011–12**

Continuum Mechanics **2 hours**

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

1 (i) The base vectors of the new Cartesian coordinates, $\mathbf{e}'_1$, $\mathbf{e}'_2$, $\mathbf{e}'_3$, are related to the base vectors of the old Cartesian coordinates, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, by $\mathbf{e}'_i = a_{ij}\mathbf{e}_j$, where a_{ij} are the elements of the transformation matrix $\hat{\mathbf{A}}$. $\mathbf{T}$ is a second-order tensor. Its components in the old Cartesian coordinates are given by $T_{ij} = \mathbf{T}(\mathbf{e}_i, \mathbf{e}_j)$. Write the expression for determining the components T'_{ij} of tensor $\mathbf{T}$ in the new Cartesian coordinates. Show that T_{ij} and T'_{ij} are related by

$$T'_{ij} = a_{ik}a_{jl}T_{kl}.$$

Show that this relation is equivalent to the matrix relation

$$\hat{\mathbf{T}}' = \hat{\mathbf{A}}\hat{\mathbf{T}}\hat{\mathbf{A}}^T,$$

where $\hat{\mathbf{A}}$ is the transformation matrix from old to new coordinates with elements a_{ij}, and $\hat{\mathbf{T}}$ and $\hat{\mathbf{T}}'$ are the matrices of components of tensor $\mathbf{T}$ in old and new coordinates respectively. ***(9 marks)***

(ii) New Cartesian coordinates, x'_1, x'_2, x'_3, are obtained from the old ones, x_1, x_2, x_3, by a rotation about the x_3-axis through an angle θ. Show that the transformation matrix is given by

$$\hat{\mathbf{A}} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(4 marks)

 Question 1 continued on next page

(iii) The components of tensor T in coordinates x_1, x_2, x_3 are given by

$$\hat{\mathsf{T}} = \begin{pmatrix} a & b & 0 \\ b & c & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Find the matrix $\hat{\mathsf{T}}'$ of components of this tensor in coordinates x_1', x_2', x_3'.
(8 marks)

(iv) You are given that $T_{ij}' = 0$ when $i \neq j$, and $a = 3$, $b = c = 1$. Determine θ.
(4 marks)

2 (i) Give the definition of a streamline. Show that, in Cartesian coordinates x_1, x_2, x_3, the system of equations for streamlines can be written in the form

$$\frac{dx_1}{d\lambda} = v_1(\mathbf{x}, t), \quad \frac{dx_2}{d\lambda} = v_2(\mathbf{x}, t), \quad \frac{dx_3}{d\lambda} = v_3(\mathbf{x}, t),$$

where $\mathbf{v} = (v_1, v_2, v_3)$ is the velocity and $\mathbf{x} = (x_1, x_2, x_3)$.
(10 marks)

(ii) The velocity field of a planar motion is given by

$$\mathbf{v} = u\left(-\frac{x_2}{a} + \frac{a x_1}{x_1^2 + x_2^2}\right)\mathbf{e}_1 + u\left(\frac{x_1}{a} + \frac{a x_2}{x_1^2 + x_2^2}\right)\mathbf{e}_2, \qquad (*)$$

where $\mathbf{e}_1$, $\mathbf{e}_2$ are the base vectors, and u and a are positive constants. Explain why, for this particular velocity field, the trajectories coincide with the streamlines.
(2 marks)

(iii) Use the variable substitution

$$x_1 = r\cos\phi, \quad x_2 = r\sin\phi$$

to obtain the system of equations defining streamlines in the x_1x_2-plane in polar coordinates r, ϕ for the velocity field $(*)$.
(6 marks)

(iv) For the velocity field $(*)$, find the equation of a streamline that contains the point of the x_1x_2-plane with polar coordinates (r_0, ϕ_0) in the form $r = r(\phi)$.
(7 marks)

3 (i) Give the definitions of the principal directions and principal stresses of the stress tensor T, and write down the equation defining the principal directions and stresses. ***(5 marks)***

(ii) Let $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ be the principal directions of the stress tensor T corresponding to the principal stresses T_1, T_2 and T_3. Show that in Cartesian coordinates with the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ the matrix $\hat{\mathsf{T}}$ of the tensor T has the diagonal form with T_1, T_2 and T_3 on the main diagonal. ***(5 marks)***

(iii) Give the expression for the surface traction $\mathbf{t}$ in terms of the stress tensor and the unit normal vector $\mathbf{n}$ to the surface. ***(2 marks)***

(iv) Consider the surfaces with the normal unit vectors

$$\mathbf{n}_1 = \frac{1}{\sqrt{3}}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3),$$

$$\mathbf{n}_2 = \frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2) - \frac{1}{\sqrt{2}}\,\mathbf{e}_3,$$

$$\mathbf{n}_3 = \frac{1}{\sqrt{6}}\,\mathbf{e}_1 - \frac{1}{\sqrt{2}}\,\mathbf{e}_2 - \frac{1}{\sqrt{3}}\,\mathbf{e}_3,$$

where $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are the unit vectors in the principal directions of T. Denote the corresponding surface tractions by $\mathbf{t}_1$, $\mathbf{t}_2$ and $\mathbf{t}_3$ respectively. You are given that $\mathbf{t}_1$ is perpendicular to $\mathbf{n}_1$, $\mathbf{t}_2$ is perpendicular to $\mathbf{n}_2$, $|\mathbf{t}_3| = 100\ \mathrm{N/m^2}$, and $T_1 > 0$. Calculate T_1, T_2 and T_3. ***(13 marks)***

4 (i) By considering the equilibrium of a small volume of water in the form of a cube, derive the expression for the water pressure p at depth h,

$$p = p_a + g\rho h,$$

where p_a is the atmospheric pressure, g the gravitational acceleration, and ρ the water density. ***(9 marks)***

(ii) There is a spherical shell made of an elastic material. The shell is filled with air at pressure $p_0 = 10^6$ N/m^2. The radius of the shell is $r_0 = 0.1$ m. At the initial moment of time the shell is above the water surface. Then a weight of mass $m = 5$ kg and of negligible volume is attached to the shell, and the shell together with the weight is immersed in water. You are given that the volume of the shell is proportional to the difference between the internal pressure of the air in the shell and the external pressure, and the pressure of the air inside the shell is inversely proportional to its volume.

(a) Show that the depth at which the shell is immersed and its radius are related by the equation

$$9\left(\frac{r}{r_0}\right)^6 + (1 + 0.1h)\left(\frac{r}{r_0}\right)^3 - 10 = 0.$$

(You can assume that the motion is quasi-static and the shell preserves its spherical shape during the motion, and take the water pressure the same at any point on the shell. You also can take $g = 10$ m/s^2, the water density $\rho = 10^3$ kg/m^3, and the atmospheric pressure $p_a = 10^5$ N/m^2.) ***(9 marks)***

(b) Derive (but do not try to solve) the equation for the depth $h(t)$ at which the shell will be at the moment of time t. Thus calculate the acceleration of the shell at the depth $h = 10$ m. (You can use without derivation Archimedes' law: the pressure force exerted on the surface of a body immersed in water is in the vertical direction, and its magnitude is equal to the weight of water displaced by the body. You also can neglect the weight of the spherical shell, and the water resistance to its motion.) ***(7 marks)***

5 You are given that, in linear elasticity, the momentum equation for the infinitesimal displacement vector $\mathbf{u}$ has the form

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2} = (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) + \mu\nabla^2\mathbf{u},$$

where $\rho = \text{const}$ is the density and λ and μ are the Lamé constants. You are also given that $\mathbf{u}$ is a planar vector, $\mathbf{u} = (w, v, 0)$, and that it depends on one spatial coordinate $x = x_1$ only, i.e. $\mathbf{u} = \mathbf{u}(t, x)$.

(i) Show that v satisfies the equation for the transverse wave,

$$\rho\frac{\partial^2 v}{\partial t^2} = \mu\frac{\partial^2 v}{\partial x^2}. \qquad (*)$$

(3 marks)

(ii) Show that equation $(*)$ has a solution in the form $v = f(t \pm x/c_s)$, where f is an arbitrary function, and find an expression for c_s in terms of ρ and μ. ***(6 marks)***

(iii) You are given that $\rho = \rho_-$, $\lambda = \lambda_-$ and $\mu = \mu_-$ in the half-space $x < 0$, while $\rho = \rho_+$, $\lambda = \lambda_+$ and $\mu = \mu_+$ in the half-space $x > 0$. You are also given that the surface traction at the surface $x = 0$ is given by

$$\mathbf{t} = \left((\lambda+2\mu)\frac{\partial u}{\partial x}, \mu\frac{\partial v}{\partial x}\right).$$

Assuming that $\mathbf{u} = (0, v, 0)$ and v depends on one spatial coordinate $x = x_1$ only, use the conditions of continuity of $\mathbf{u}$ and $\mathbf{t}$ at $x = 0$ to derive two boundary conditions for v at $x = 0$. ***(5 marks)***

(iv) There is a plane harmonic transverse wave propagating in the half-space $x < 0$ in the positive x-direction. This wave is defined by $v = a\cos[\omega(t - x/c_{s-})]$. Explain why we can look for the solution to equation $(*)$ in the form

$$v = \begin{cases} a\cos[\omega(t - x/c_{s-})] + b\cos[\omega(t + x/c_{s-})], & x < 0, \\ c\cos[\omega(t - x/c_{s+})], & x > 0. \end{cases}$$

Use the boundary consitions at $x = 0$ to calculate b/a and c/a and find the coefficient of reflection $R = b^2/a^2$. ***(11 marks)***

End of Question Paper