The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2011–2012**

MAS271 Methods for differential Equations **2 hours**

Answer all four questions.

1 (i) Solve the equation

$$(1-t^2)x'' - 6tx' - 4x = 0$$

near the ordinary point $t = 0$. Give the general solution.
Using the ratio test, find the radius of convergence.

(15 marks)

(ii) Do the following systems have equilibrium points? If so, find the equilibrium points and investigate the nature of the stability of the system at these points.

(a) $\dot{x} = x(2-x), \quad \dot{y} = -y + x$;

(b) $\dot{x} = \dfrac{1}{y}, \quad \dot{y} = \dfrac{2}{x}$.

(10 marks)

2 (i) Let

$$V = y^2 + \omega^2 x^2$$

and

$$\dot{x} = y, \;\; \dot{y} = -cy - \omega^2 x \;\; (\text{where } c > 0).$$

Deduce whether V is a weak or a strong Liapunov function and using this, comment on the stability of the point (0,0).

(6 marks)

 Question 2 continued on next page

2 (continued)

(ii) Show that the first integral of the differential equation

$$\ddot{x} + 4x = 0$$

is

$$4x^2 + y^2 = \text{constant},$$

where $y = \dot{x}$.

Sketch these trajectories in the xy-plane, indicating the direction of time, t, increasing.
What can you conclude about the nature of the equilibrium point (0,0)? Justify your answer. ***(8 marks)***

(iii) Give general solution for $x > 0$:

(i) $x^2y^{''} + 5xy^{'} + 4y = 0$,

(ii) $x^2y^{''} + xy^{''} + y = 0$.

(11 marks)

3 (i) Find the power series solution of

$$x^{''} - tx^{'} + x = 0$$

with $x(0) = 1$ and $x^{'}(0) = 0$. ***(7 marks)***

(ii) The system of three equations is given by

$$(x, y, z)^{'} = (4x - y,\ 3x + y - z,\ x + z).$$

express it in the vector-matrix form. ***(3 marks)***

(iii) Deduce whether the following Liapunov functions are positive definite or negative definite:

(a) $4x^2 + 3xy + 2y^2$,

(b) $-3x^2 - 4xy - y^2$. ***(6 marks)***

(iv) Find and classify the equilibrium points of the system

$$\dot{x} = x(\mu - x),$$

where μ is a control parameter. Sketch the stable and unstable regions by using solid and dashed lines respectively. Explain what is meant by a *transcritical bifurcation*. ***(9 marks)***

4 (i) Show that the substitution $z(x) = y(x)\sqrt{x}$ renders Bessel's equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0; \ \ \nu \geq 0,$$

in the form

$$z'' + \left(1 + \frac{1 - 4\nu^2}{4x^2}\right) z = 0,$$

where ν is a constant.

(8 marks)

For $x \gg 1$, the above equation can be approximated as

$$z'' + z = 0.$$

Write down the general solution. ***(3 marks)***

By substituting $y(x) = z(x)/\sqrt{x}$, write the general solution for y.

(2 marks)

(ii) For the following equations find the equilibrium points, investigate the nature of stability at these points and sketch the phase portrait for the stable equilibrium point:

$$\dot{x} = 3x - x^2 - 2xy, \quad \dot{y} = -y + xy$$

(12 marks)

End of Question Paper