The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2011–12**

Continuity and Integration **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) Give the formal definition of the notion of a sequence of real numbers *converging to a limit.* ***(2 marks)***

Use the definition to show that the sequence $\frac{1}{n^2} \to 0$. ***(5 marks)***

(ii) State the sandwich rule and use it to show that the sequence $\left(\frac{\sin^2 n}{n^2}\right)$ converges to 0. ***(4 marks)***

(iii) Let (x_n) be the sequence given by the recurrence $x_{n+1} := \sqrt{5x_n}$ for all $n \geq 1$ and first term $x_1 := 1$.

(a) Show that $\frac{x_{n+1}}{x_n} = \sqrt{\frac{x_n}{x_{n-1}}}$ for $n \geq 2$ and deduce that (x_n) is an increasing sequence.

(b) Show that (x_n) is bounded above by 5 and deduce that (x_n) converges to a limit.

(c) Find the limit of (x_n). ***(9 marks)***

2 (i) Prove that if E and F are non-empty sets of real numbers which are bounded below then $\inf(E \cup F)$ is the minimum of $\inf E$ and $\inf F$.

(8 marks)

(ii) State which of the statements below are true and which are false. Prove those that are true, and provide counter examples for those that are false. Theorems proved in lectures may be used without proof, provided they are precisely stated.

(a) If a set of real numbers has a maximum and a minimum then it must be finite.

(b) If E is a bounded non-empty set of rational numbers then $\sup E$ is a rational number.

(c) There is a set for which -2012 is an upper bound and 2012 is a lower bound.

(d) An increasing sequence of negative numbers converges to a negative limit.

(e) If $E \subset F$ are non-empty and bounded then $\sup E \leq \sup F$.

(12 marks)

3 (i) Define what it means for a real-valued function f to be *continuous* at a point a in its domain. *(2 marks)*

For each of the following, give an example of a function with the stated property. You do not have to prove that your function has the required property.

(a) A *bounded discontinuous* function $f : [-1, 1] \longrightarrow \mathbb{R}$ whose restriction to $[-1, 0) \cup (0, 1]$ is continuous.

(b) A *bounded continuous* function $f : [0, 2012) \longrightarrow \mathbb{R}$ which has no maximum.

(c) A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which is discontinuous everywhere.

(6 marks)

(ii) State the Intermediate Value Theorem. *(2 marks)*

Let $f, g : [a, b] \longrightarrow \mathbb{R}$ be two continuous functions such that $f(a) < g(a)$ and $f(b) > g(b)$. Show that there is a $c \in (a, b)$ such that $f(c) = g(c)$.

(10 marks)

4 (i) Describe what it means for a function f to be *differentiable* at some point a in its domain. ***(2 marks)***

Use the definition to find the derivative of x^2. ***(3 marks)***

(ii) State Rolle's Theorem. ***(2 marks)***

Suppose that $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ are continuous functions which are differentiable on (a,b). Assume that there is **no** $t \in (a,b)$ with $g'(t) = 0$.

(a) Prove that if $a < c \leq b$ then $g(c) \neq g(a)$.

(b) Consider the function $h(t) := \big(f(b) - f(a)\big)g(t) - \big(g(b) - g(a)\big)f(t)$. Use Rolle's Theorem, carefully explaining your reasoning, to show that there is a $c \in (a,b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

(13 marks)

5 (i) Explain what it means to say that a bounded function $f : [a,b] \longrightarrow \mathbb{R}$ is *Riemann integrable* on the interval $[a,b]$. ***(6 marks)***

(ii) Let $f : [a,b] \longrightarrow \mathbb{R}$ be a bounded function. Suppose there is a sequence of partitions P_n of $[a,b]$ such that $U_{P_n}(f) - L_{P_n}(f) \to 0$. Prove that f is Riemann integrable on $[a,b]$. ***(4 marks)***

(iii) Prove that if $f : [0,1] \longrightarrow \mathbb{R}$ is a decreasing function then f is Riemann integrable on $[0,1]$. ***(10 marks)***

End of Question Paper