The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2011-2012

NUMERICAL LINEAR ALGEBRA — 2 hours

Answer FOUR questions. You are advised not to answer more than four questions: if you do, only your best four will be counted.

1 (i) Given a set of data (x_j, f_j), $j = 0, 1, ..., m$ and the basis functions $\phi_0(x)$, $\phi_1(x)$, ..., $\phi_n(x)$, we use

$$Y(x) = \sum_{i=0}^{n} a_i \phi_i(x)$$

to approximate the data in the least-squares sense, where a_i, $i = 0, 1, ..., n$ are the coefficients. Show that the normal equations for the coefficients are

$$\sum_{i=0}^{n} \left[a_i \sum_{j=0}^{m} \phi_i(x_j)\phi_k(x_j) \right] = \sum_{j=0}^{m} f_j \phi_k(x_j),$$

for $k = 0, 1, 2, ..., n$.

(10 marks)

(ii) Using a suitable transformation, determine a least-squares fit of the form $Y(x) = A + \ln(B + x)$ to the data:

j :	0	1	2	3
x_j :	1.5	2.5	3.5	4.5
f_j :	2.9263	3.2328	3.7041	3.2007

where A and B are the coefficients to be determined.

(15 marks)

2 (i) Write down the definition of the condition number $\mathcal{K}(A)$ of a matrix A.

(2 marks)

(ii) Consider a system of linear algebraic equations $A\mathbf{x} = \mathbf{b}$, for which $\mathbf{x}$ is the exact solution. The residual vector $\mathbf{r}$ is defined by $\mathbf{r} = A\hat{\mathbf{x}} - \mathbf{b}$, where $\hat{\mathbf{x}}$ is an approximate solution. Show that

$$\frac{||\hat{\mathbf{x}} - \mathbf{x}||}{||\mathbf{x}||} \leq \mathcal{K}(A)\frac{||\mathbf{r}||}{||\mathbf{b}||}.$$

(12 marks)

(iii) Consider a system where

$$A = \begin{bmatrix} 449 & 30 \\ 30 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 4.79 \\ 0.31 \end{bmatrix}.$$

If the residual error $\mathbf{r} = (0, -0.02)^T$, estimate the bound of the relative error in the approximate solution $\hat{\mathbf{x}}$ using the **2-norm**. Can we ascertain that $\hat{\mathbf{x}}$ is a good approximation? (Hint: If the eigenvalues of a real, symmetric matrix A are λ_1, λ_2, ...,λ_n, where $|\lambda_1| \geq |\lambda_2| \geq ... \geq |\lambda_n|$, then $\mathcal{K}_2(A) = |\lambda_1|/|\lambda_n|$.)

(11 marks)

3 (i) The Householder reflection matrix is given by

$$P = \left(I - 2\frac{\mathbf{w}\mathbf{w}^T}{\mathbf{w}^T\mathbf{w}}\right),$$

where $\mathbf{w}$ is a $(m+1)$ dimensional column vector. Show that P is an orthogonal matrix.

(4 marks)

(ii) Given the vector $\mathbf{x}^T = (x_0, x_1, ..., x_m)$, define $\mathbf{w}$ by

$$\mathbf{w} = \mathbf{x} + sign(x_0)\,||\mathbf{x}||_2\,\mathbf{e}_0,$$

where $\mathbf{e}_0$ is the first column of the $(m+1) \times (m+1)$ identity matrix. Let $\hat{\mathbf{x}} = P\mathbf{x}$, where P is defined in Part (i) with $\mathbf{w}$ given above, show that

$$\hat{\mathbf{x}} = -sign(x_0)||\mathbf{x}||_2\,\mathbf{e}_0.$$

(6 marks)

(iii) Show that $||\hat{\mathbf{x}}||_2 = ||\mathbf{x}||_2$.

(3 marks)

(iv) Apply two Householder reflections to transform the matrix

$$A = \begin{bmatrix} 1 & 3.1 \\ 1 & 3.3 \\ 1 & 3.5 \\ 1 & 3.7 \end{bmatrix}$$

to the row echelon form.

(12 marks)

4 (i) The matrix A has eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ satisfying

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq ... \geq |\lambda_n| > 0$$

with linearly independent eigenvectors $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n$. The eigenvectors have been normalized so that the largest element of each one is unity.

(a) Write down the power iteration for finding the dominant eigenvalue of A and its eigenvector. ***(3 marks)***

(b) Prove that the iteration converges to the dominant eigenvalue and its eigenvector. ***(11 marks)***

(ii) Given matrix

$$A = \begin{pmatrix} 6.0 & 3.4 & 1.3 \\ 3.4 & 7.0 & 2.0 \\ 1.3 & 2.0 & 2.3 \end{pmatrix},$$

use the power method to find the estimate for the dominant eigenvalue and its eigenvector. Start with $\mathbf{z}_0^T = (0, -0.2, 1.0)$, and perform **two** iterations. State clearly the approximate values for the eigenvalue at each iteration. Work correct to four decimal places. ***(7 marks)***

(iii) Describe a method to find the eigenvalue of A that is the closest to a given number p. State your reasons. Do **not** try to find the eigenvalue. ***(4 marks)***

5 (i) Show that $\rho(A) \leq ||A||$ for any matrix A, where $\rho(A)$ is the spectral radius of A. ***(5 marks)***

(ii) A linear system $A\mathbf{x} = \mathbf{b}$ can be rearranged to the form $\mathbf{x} = H\mathbf{x} + \mathbf{d}$, from which one can define the iteration

$$\mathbf{x}^{(k+1)} = H\mathbf{x}^{(k)} + \mathbf{d}$$

where H is an $n \times n$ matrix and $\mathbf{d}$ is an $n \times 1$ column vector of known constant values. Given that the iteration converges if and only if $\rho(H) < 1$, show that the iteration converges when $\|H\| < 1$. ***(2 marks)***

(iii) Use the Jacobi iterative method to obtain **two** successive approximations to the solution of the system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{pmatrix} 10 & 4 & 0 \\ 4 & 5 & 3 \\ 0 & 3 & 5 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix},$$

using $\mathbf{x}^{(0)} = (1, 1, 1)^T$ as the starting vector and working correct to four decimal places. Then show that the Jacobi iteration converges. ***(18 marks)***

End of Question Paper