The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2011–12**

Differential and difference equations **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) Solve the separable differential equation

$$(x+2)y' - xy = 0,$$

subject to the condition $y(0) = 1$. ***(12 marks)***

(ii) An electric circuit has a resistance (R) and inductance (L) connected in series and attached to a voltage source. The current flowing in the circuit is given by the differential equation

$$L\frac{dI}{dt} + RI = \sin(2t).$$

Use the integrating factor technique to find the value of the current, $I = I(t)$, subject to the initial condition $I(0) = 0$. ***(13 marks)***

2 (i) The rate of growth, P, of certain crabs in a sea shore satisfies the equation

$$\frac{dP}{dt} = k(650 - P),$$

where k is a constant. Initially the population of crabs was 300 and it grew to 500 over a period of 2 years. Find the population equation and estimate the crab population for the third year. What happens with the population, P, when $t \to \infty$? ***(12 marks)***

(ii) The population of two species of birds competing for the same kind of insects are given by the system

$$\frac{dx}{dt} = 2x(1-x) - 2xy, \qquad \frac{dy}{dt} = y(1-2y) - \alpha xy$$

where α is a positive constant.

 Question 2 continued on next page

2 (continued)

(a) Find all equilibrium levels of the system. ***(8 marks)***

(b) Show that when $\alpha > 1$ the only two equilibrium states for which at least one population is non-zero are $x = 0$, $y = 1/2$ and $x = 1$, $y = 0$. ***(5 marks)***

3 (i) The equation of a simple harmonic oscillator is

$$\frac{d^2x}{dt^2} = -\omega^2 x, \tag{1}$$

where $x = x(t)$ and ω is the angular frequency of oscillations. Find a particular solution of the equation (1) which satisfies the initial conditions

$$x(0) = 1, \quad x'(0) = 0.$$

(5 marks)

(ii) In the case of forced oscillations, equation (1) is modified as

$$\frac{d^2x}{dt^2} + \omega^2 x = F_0 \cos \Omega t. \tag{2}$$

Find the solution of this equation subject to the initial conditions $x(0) = x_0$ and $x'(0) = 0$. ***(10 marks)***

(iii) For the same initial conditions as in part (*ii*), find the solution of the differential equation (2) when the two frequencies are equal, i.e. $\omega = \Omega$, (often called *resonance*) and describe the behaviour of the amplitude of the solution when $t \to \infty$. ***(10 marks)***

4 (i) Solve the second order difference equation

$$u_{n+2} + u_{n+1} + \frac{1}{4}u_n = n^2 + 5 \times 2^n,$$

subject to the conditions $u_0 = 1$ and $u_1 = 1$. ***(18 marks)***

(ii) Solve the first order difference equation

$$u_{n+1} + \frac{1}{2}u_n = \frac{3}{2},$$

subject to the condition $u_0 = 1/2$. ***(7 marks)***

End of Question Paper