Data provided: Formula Sheet

MAS420

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2011–12**

MAS420 Signal Processing **2 hours**

Answer ***four*** *questions. You are advised* ***not*** *to answer more than four questions; if you do, only your best four will be counted.*

1 (i) In the Hilbert space of finite power signals of period T with inner product

$$(f,g) = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)g^*(t)dt,$$

prove that the set $\phi_n(t) = e^{in\sigma t} : -\infty < n < \infty$, where $\sigma = \dfrac{2\pi}{T}$, forms an orthonormal basis (both orthogonality and unit norm conditions must be proved). ***(4 marks)***

(ii) If $f(t) = \sum_{n=-\infty}^{\infty} c_n\phi_n(t)$, use orthonormality to prove that the power in $f(t)$ can be written

$$P = \sum_{n=-\infty}^{\infty} |c_n|^2$$

(4 marks)

(iii) Using the frequency shift property of Fourier transforms, show that, if $f(t) \leftrightarrow F(\omega)$, then

$$f(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}[F(\omega-\omega_0) + F(\omega+\omega_0)].$$

Using this result (or otherwise), show that

$$g(t) = p_{\frac{\pi}{2}}(t)\cos t \leftrightarrow \frac{\pi}{2}\left[\operatorname{sinc}\left(\frac{\pi}{2}(\omega-1)\right) + \operatorname{sinc}\left(\frac{\pi}{2}(\omega+1)\right)\right]$$

(4 marks)

(iv) Sketch a few periods (centred on $t=0$) of the signal

$$g_\pi(t) = \sum_{n=-\infty}^{\infty} g(t-n\pi)$$

where $g(t)$ is the signal defined in part (iii). Using the fact that a periodic signal of period T, $f_T(t)$, has complex Fourier coefficients $c_n = \dfrac{1}{T}F(n\sigma)$, where $F(\omega)$ is the Fourier transform of $f_T(t)p_{\frac{T}{2}}(t)$ and $\sigma = \dfrac{2\pi}{T}$, show that $g_\pi(t)$ has complex Fourier coefficients given by

$$c_n = \frac{2}{\pi}\frac{(-1)^{n+1}}{(4n^2-1)}$$

(6 marks)

(v) The signal $g_\pi(t)$ is passed through an ideal low-pass filter, $p_3(\omega)$, to yield a signal $h(t)$. Write $h(t)$ as a sine/cosine series and show that $\sim 99\%$ of the power in $g_\pi(t)$ is passed through this filter. ***(7 marks)***

2 (i) Prove the convolution theorem

$$f * g(t) \longleftrightarrow F(\omega)G(\omega)$$

where $f(t)$ and $g(t)$ are signals with Fourier transforms $F(\omega)$ and $G(\omega)$ respectively. ***(4 marks)***

(ii) Prove that a system whose operation is described by a convolution with an impulse response function, $h(t)$, is linear and shift invariant. ***(4 marks)***

(iii) For a system with impulse response function $h(t) = e^{-t}U(t)$, show, with the aid of clear diagrams, that the response, $g(t)$, to the input signal $p_1(t)$ is given by

$$g(t) = \begin{cases} (e - e^{-1})e^{-t} & t \geq 1 \\ (1 - e^{-(1+t)}) & |t| \leq 1 \\ 0 & t < -1. \end{cases}$$

Find the Fourier transform of $g(t)$ and hence evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\operatorname{sinc}\omega}{1 + i\omega} d\omega.$

(11 marks)

(iv) Evaluate and sketch the phase and amplitude spectra of the signal $g(t)$ found in part (iii). ***(6 marks)***

3 (i) Define the equivalent rectangle resolution, τ, of a signal, $f(t)$, stating clearly the conditions under which it is defined. Explain, using a clear diagram, how it is related to the signal $f(t)$. ***(6 marks)***

(ii) The time-bandwidth theorem for a $\Omega-$bandlimited signal for which the equivalent rectangle resolution τ is defined states that $\Omega\tau \geq \pi$. Use the Cauchy-Schwarz inequality to prove the theorem including the conditions needed to achieve the minimum possible time-bandwidth product. ***(7 marks)***

(iii) Using the property $\cos\omega_0 t = \dfrac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2}$, show, with the aid of clear diagrams, that the signal

$$f(t) = \mathrm{sinc}(\alpha t)\cos(\omega_0 t),$$

where $\alpha > 0$ and $\omega_0 > 0$, has energy

$$E(f) = \begin{cases} \dfrac{\pi}{\alpha^2}\left(\alpha - \dfrac{\omega_0}{2}\right) & \omega_0 < \alpha \\[2ex] \dfrac{\pi}{2\alpha} & \alpha \leq \omega_0 \end{cases}$$

Verify that $f(t)$ satisfies the conditions of the time-bandwidth theorem. In the case $\alpha < \omega_0$ calculate the bandwidth Ω and the equivalent rectangle resolution, τ, and hence verify that $\Omega\tau > \pi$. ***(12 marks)***

4 (i) Define the the system transfer function (STF), without any reference to the Fourier transform or the impulse response function;

Write down the expression relating the Fourier transform, $G(\omega)$, of the output of a system with STF $H(\omega)$ to the Fourier transform, $F(\omega)$, of the input. ***(3 marks)***

(ii) If a LSI system has a STF, $H(\omega)$, with a real impulse response function, $h(t)$, show that the response to the input $\cos\omega_0 t$ is $Re(H(\omega_0)e^{i\omega_0 t})$, and write down the corresponding response to input $\sin\omega_0 t$. ***(3 marks)***

(iii) A system S acts as a differentiator,

$$S(f) = \frac{df}{dt}.$$

Show that S is a LSI system with system transfer function $H(\omega) = i\omega$. Hence show that

$$\frac{d}{dt}(f * g) = \frac{df}{dt} * g = f * \frac{dg}{dt}.$$

(8 marks)

(iv) A system T acts on its input by convolution with a Gaussian followed by differentiation: if the input is $f(t)$ then the output is

$$g(t) = \frac{d}{dt}\left(f(t) * e^{-t^2/4}\right).$$

Find the system transfer function for T and calculate the response of the system to the input

$$f(t) = 1 + \frac{1}{2}\sin 2t - \cos t.$$

(8 marks)

(v) Show that the system T of part (iv) is equivalent to a system whose response to the *general* input $f(t)$ is

$$g(t) = -\frac{1}{2}f(t) * te^{-t^2/4}.$$

(3 marks)

5 (i) Assuming the Fourier transform pair $\overline{\delta}_T(t) \longleftrightarrow \sigma\overline{\delta}_\sigma(\omega)$, where $\overline{\delta}_T(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$, $\overline{\delta}_\sigma(\omega)$ is defined similarly and $\sigma = 2\pi/T$, prove that

$$f_s(t) \equiv f(t)\overline{\delta}_T(t) \longleftrightarrow \frac{1}{T}\sum_{n=-\infty}^{\infty} F(\omega - n\sigma) \equiv F_s(\omega),$$

where $F(\omega)$ is the Fourier transform of $f(t)$.

Show also that $F_s(\omega) = \sum_{n=-\infty}^{\infty} f(nT)e^{-i\omega nT}$.

(6 marks)

(ii) $g(t)$ is obtained from the sampled signal $f_s(t)$ by the sinc interpolation formula

$$g(t) = \sum_{n=-\infty}^{\infty} f(nT)\,\mathrm{sinc}\left\{\frac{\sigma}{2}(t-nT)\right\}.$$

Prove that

$$G(\omega) = p_{\sigma/2}(\omega)\sum_{n=-\infty}^{\infty} F(\omega - n\sigma)$$

where $G(\omega)$ is the Fourier transform of $g(t)$. State the conditions under which $g(t) = f(t)$. Clear diagrams are likely to help your answer.

(5 marks)

(iii) Find the Nyquist frequency, in Hz, of the signal

$$f(t) = \mathrm{sinc}^2(4t).$$

(2 marks)

(iv) This signal is sampled at 3/4 of its Nyquist frequency and the samples are used to form a signal $g(t)$ by sinc interpolation. Making use of clear diagrams, find $G(\omega)$ and hence $g(t)$.

Verify that $g(t) \neq f(t)$ by considering the values of the two signals at $t = \pi/4$.

(8 marks)

(v) Using the relation $g(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)d\omega$, show that for any signal, $f(t)$, and any sample spacing, T, the signal $g(t)$ defined in part (ii) satisfies $g(0) = f(0)$.

(4 marks)

End of Question Paper

Formula sheet

Function Definitions:

Rectangular pulse:

$$p_a(t) = \begin{cases} 1 & |t| \le a \\ 0 & |t| > a \end{cases}$$

Triangular pulse:

$$q_a(t) = \begin{cases} 1 - |t|/a & |t| \le a \\ 0 & |t| > a \end{cases}$$

Step function:

$$U(t) = \begin{cases} 1 & t \ge 0 \\ 0 & t < 0 \end{cases}$$

Fourier Transform Pairs:

$$\begin{aligned}
p_a(t) &\longleftrightarrow 2a\,\mathrm{sinc}(a\omega) \\
q_a(t) &\longleftrightarrow a\,\mathrm{sinc}^2(a\omega/2) \\
\mathrm{sinc}(at) &\longleftrightarrow \frac{\pi}{a} p_a(\omega) \\
\mathrm{sinc}^2(at) &\longleftrightarrow \frac{\pi}{a} q_{2a}(\omega) \\
\mathrm{e}^{-at}U(t) &\longleftrightarrow \frac{1}{a + i\omega} \\
\delta(t) &\longleftrightarrow 1 \\
\delta(t - t_0) &\longleftrightarrow \mathrm{e}^{-i\omega t_0} \\
1 &\longleftrightarrow 2\pi\delta(\omega) \\
\mathrm{e}^{i\omega_0 t} &\longleftrightarrow 2\pi\delta(\omega - \omega_0) \\
\mathrm{e}^{-t^2/2\sigma^2} &\longleftrightarrow \sigma\sqrt{2\pi}\mathrm{e}^{-\sigma^2\omega^2/2}
\end{aligned}$$

Fourier transform:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-i\omega t}\,\mathrm{d}t$$

Inverse Fourier transform:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\mathrm{e}^{i\omega t}\,\mathrm{d}\omega$$

Duality theorem: If $f(t) \longleftrightarrow F(\omega)$ then $F(t) \longleftrightarrow 2\pi f(-\omega)$

Scaling: If $f(t) \longleftrightarrow F(\omega)$ then $f(at) \longleftrightarrow \frac{1}{|a|}F(\omega/a)$.

Translation: If $f(t) \longleftrightarrow F(\omega)$ then $f(t - t_0) \longleftrightarrow \mathrm{e}^{-i\omega t_0}F(\omega)$.

Frequency Shift: If $f(t) \longleftrightarrow F(\omega)$ then $\mathrm{e}^{i\omega_0 t}f(t) \longleftrightarrow F(\omega - \omega_0)$

Fourier Series: If $f_T(t)$ is periodic with period T then, with $\sigma = 2\pi/T$, the complex Fourier series is

$$f_T(t) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{in\sigma t}$$

where

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f_T(t)\mathrm{e}^{-in\sigma t}\ \mathrm{d}t$$

Likewise, the real Fourier series is

$$f_T(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\sigma t + b_n \sin n\sigma t)$$

where

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f_T(t) \cos n\sigma t\ \mathrm{d}t$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f_T(t) \sin n\sigma t\ \mathrm{d}t$$

Parseval's Theorem: If V is a Hilbert space, $\{\phi_n\}$ is an orthonormal basis for V and $f = \sum_n c_n\phi_n$, then

$$\|f\|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Plancherel's Theorem: If $f(t) \longleftrightarrow F(\omega)$ and $g(t) \longleftrightarrow G(\omega)$ then

$$\int_{-\infty}^{\infty} f(t)g^*(t)\ \mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)G^*(\omega)\ \mathrm{d}\omega$$

Energy Theorem: If $f(t) \longleftrightarrow F(\omega)$ then

$$\int_{-\infty}^{\infty} |f(t)|^2\ \mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\ \mathrm{d}\omega$$

Convolution Theorem: If $f(t) \longleftrightarrow F(\omega)$ and $g(t) \longleftrightarrow G(\omega)$ then

$$f * g(t) = \int_{-\infty}^{\infty} f(s)g(t-s)\ \mathrm{d}s \longleftrightarrow F(\omega)G(\omega)$$

Product Theorem: If $f(t) \longleftrightarrow F(\omega)$ and $g(t) \longleftrightarrow G(\omega)$ then

$$f(t)g(t) \longleftrightarrow \frac{1}{2\pi}F(\omega) * G(\omega).$$