The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring semester 2011

Stochastic Processes and Finance — **3 hours**

Candidates may bring to the examination a calculator that conforms to University regulations.
Full marks may be obtained by complete answers to five questions. All answers will be marked, but credit will be given only for the best five answers. Total marks 100.

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1 (i) Let $(X_n, n \in \mathbb{Z}_+)$ be a (discrete-parameter) stochastic process which is defined on a probability space $(\Omega, \mathcal{F}, P)$ that is equipped with a filtration $(\mathcal{F}_n, n \in \mathbb{Z}_+)$. Suppose that the process is adapted to the filtration and also integrable. Explain precisely what it means for $(X_n, n \in \mathbb{Z}_+)$ to be

(a) a martingale, ***(1 mark)***

(b) a submartingale. ***(1 mark)***

(ii) Suppose that $(X_n, n \in \mathbb{Z}_+)$ is a martingale and that each $\mathbb{E}(X_n^2) < \infty$. Show that $(X_n^2, n \in \mathbb{Z}_+)$ is a submartingale with respect to the same filtration. [Hint: First show that $\mathbb{E}((X_n - X_{n-1})^2 | \mathcal{F}_{n-1}) = \mathbb{E}(X_n^2 | \mathcal{F}_{n-1}) - X_{n-1}^2$.] ***(4 marks)***

(iii) Let $(U_n, n \in \mathbb{Z}_+)$ and $(W_n, n \in \mathbb{Z}_+)$ be martingales defined on the same probability space. Define a new process $(Z_n, n \in \mathbb{Z}_+)$ by

$$Z_n = U_n + W_n.$$

(a) Show that $(Z_n, n \in \mathbb{Z}_+)$ is a martingale ***(4 marks)***

(b) If $\mathbb{E}(U_1) = 3$ and $\mathbb{E}(W_2) = 5$ what can you say about $\mathbb{E}(Z_3)$? Justify your answer. ***(2 marks)***

(iv) (a) Explain what it means for a stochastic process $(A_n, n \in \mathbb{N})$ to be *previsible* with respect to the filtration $(\mathcal{F}_n, n \in \mathbb{Z}_+)$. ***(1 mark)***

(b) Let T be a $(\mathcal{F}_n, n \in \mathbb{Z}_+)$ stopping time. Explain why the process defined by $A_n = 1_{\{T<n\}}$ is previsible. ***(3 marks)***

(c) If $(X_n, n \in \mathbb{Z}_+)$ is a martingale and $(A_n, n \in \mathbb{N})$ is previsible, deduce that $(Y_n, n \in \mathbb{N})$ satisfies the martingale property where

$$Y_n = \sum_{i=1}^{n} A_i (X_i - X_{i-1}).$$

(4 marks)

2 Throughout this question $(B(t), t \geq 0)$ is a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$ which is equipped with the natural filtration $(\mathcal{F}_t, t \geq 0)$ generated by the Brownian motion.

(i) Let Y be a $\mathcal{F}_1$-measurable random variable for which $\mathbb{E}(Y^2) = 2$. Define a stochastic process $(H(t), t \geq 0)$ as follows:

$$H(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 1 \\ Y & \text{if } 1 < t \leq 2 \\ -\dfrac{Y}{2} & \text{if } 2 < t \leq 4 \\ 0 & \text{if } t > 4. \end{cases}$$

(a) Write $H(t)$ explicitly as a sum involving the random variable Y and appropriate indicator functions. ***(2 marks)***

(b) Calculate $\displaystyle\int_0^4 \mathbb{E}(H(s)^2)ds$ ***(4 marks)***

(c) Use the result of (a) to write the stochastic integral $I = \displaystyle\int_0^4 H(s)dB(s)$ as a sum of three random variables. ***(2 marks)***

(d) Use the result of (c) to calculate $\mathbb{E}(I^2)$. What general feature of stochastic integrals do the results of (b) and (d) illustrate? ***(4 marks)***

(ii) Find the stochastic differentials of the process $(Y(t), t \geq 0)$ where for each $t \geq 0, Y(t) = e^{-\cos(B(t))}$. ***(3 marks)***

(iii) Find the unique value of α for which the process $(M(t), t \geq 0)$ is a martingale where

$$M(t) = e^{\alpha t} \sin(2B(t)).$$

(5 marks)

3 Throughout this question $(B(t), t \geq 0)$ is a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$.

(i) Write down the solution of the stochastic differential equation (SDE) :

$$dY(t) = 12\cos(3t)dt + 5dB(t),$$

with initial condition $Y(0) = 4$. ***(2 marks)***

(ii) Consider a general linear SDE of the form

$$dY(t) = \sigma(t)dB(t) + (c_1(t)Y(t) + c_2(t))dt,$$

where σ, c_1 and c_2 are suitable deterministic functions of time. Show that the unique solution to this equation, with initial condition $Y(0) = Y_0$, is given by

$$Y(t) = \rho(t)^{-1}\left[Y_0 + \int_0^t \sigma(s)\rho(s)dB(s) + \int_0^t c_2(s)\rho(s)ds\right],$$

where $\rho(t) = \exp\left\{-\int_0^t c_1(s)ds\right\}$.

[Hint: Apply Itô's product formula to the process $X(t) = Y(t)\rho(t)$.] ***(5 marks)***

Use this result to solve the following SDEs:

(a)

$$dY(t) = -6Y(t)dt + 13dB(t),$$

with $Y_0 = 5$. ***(1 mark)***

(b)

$$dY(t) = e^{-t}dB(t) + (\sin(t)Y(t) + \cos(t))dt,$$

with $Y_0 = 1$. ***(3 marks)***

In the case of (a) write down the infinitesimal mean and variance and the associated partial differential equation. ***(3 marks)***

(iii) Use the substitution $X(t) = \log_e(Y(t))$ and the result of (ii) to find the unique solution of

$$dY(t) = -3\log_e(Y(t))Y(t)dt + 2Y(t)dB(t),$$

with $Y_0 = 5$. ***(6 marks)***

4 (i) Consider a finite market model based on a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $(\mathcal{F}_n, n \in \mathcal{T})$ where $\mathcal{T} = \{0, 1, \ldots, T\}$. The market comprises two financial assets S_0 and S_1 where S_0 is the *numéraire*. Explain what is meant by a martingale measure P^* defined on $(\Omega, \mathcal{F})$. If P^* exists, what can you say about the market? What more can you say about the market if P^* not only exists but is also unique? ***(3 marks)***

(ii) Consider the binomial asset pricing model. Here there are two financial assets S_0 and S_1 as above. The numéraire S_0 is a risk-free investment with interest rate $r > 0$ so each $S_0(n) = (1 + r)^n$ while $S_1(n) = S_1(n - 1)L(n)$ where the $L(n)$s are i.i.d. Bernoulli random variables each taking the value x with probability p and the value y with probability $1 - p$.

(a) Explain the significance of the inequalities $x < r + 1 < y$. If these inequalities hold, what is the significance of the quantity $q = \dfrac{r + 1 - x}{y - x}$? ***(2 marks)***

(b) Assume that the market is complete and find an expression for the hedging portfolio which replicates a European contingent claim of the form $X = f(S(T))$. ***(6 marks)***

Hint: The arbitrage price of the option X at time n is

$$\pi_X(n) = (1 + r)^{-(T-n)} F_{T-n}(S(n), q)$$

where

$$F_n(a, q) = \sum_{r=0}^{n} \binom{n}{r} q^r (1 - q)^{n-r} f(a x^r y^{n-r}),$$

for each $a \in \mathbb{R}$.

(iii) (a) Let $Y = (Y(n), n \in \mathcal{T})$ be an adapted integrable process. Explain what is meant by the *Snell envelope* of the process Y. Is the Snell envelope a submartingale, a supermartingale or both? Justify your answer. ***(3 marks)***

(b) Let $X = (X(n), n \in \mathcal{T})$ be the pay-off process associated with an American contingent claim (ACC) with terminal time T. Explain carefully how the Snell envelope concept is related to the price process of the option. ***(4 marks)***

(c) What is the *optimal stopping problem*? Explain how the Snell envelope is used to solve this. ***(2 marks)***

5 Let $B = (B(t), 0 \leq T)$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$. Throughout this question you may take the filtration $(\mathcal{F}_t, 0 \leq t \leq T)$ to be the one that is generated by this Brownian motion so that each $\mathcal{F}_t = \sigma\{B(s), 0 \leq s \leq t\}$.

In the standard Black-Scholes model, the price $S(t)$ at time t of a stock is given by a geometric Brownian motion

$$S(t) = s\exp\{\sigma B(t) + \mu t\},$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$. We assume that there is a risk-free investment whose value at time t is $A(t) = e^{rt}$, where $r > 0$ is the interest rate.

(i) Find the stochastic differential equation (SDE) which is satisfied by the stock price. ***(3 marks)***

(ii) Find the SDE which is satisfied by the discounted stock price $\widetilde{S}(t) = A(t)^{-1}S(t)$. ***(2 marks)***

(iii) From now on suppose that the stock pays a continuous dividend at a fixed rate c where $0 < c < 1$. Write down the form of the SDEs satisfied by the tradeable asset $Z(t) = e^{ct}S(t)$ and by its discounted form $\widetilde{Z}(t) = e^{-rt}Z(t)$. ***(2 marks)***

(iv) The original measure P is changed to the equivalent probability measure Q where

$$\frac{dQ}{dP} = \exp\left\{\int_0^T F(s)dB(s) - \frac{1}{2}\int_0^T F(s)^2 ds\right\},$$

and Girsanov's theorem then tells us that $(W(t), 0 \leq t \leq T)$ is a Brownian motion on $(\Omega, \mathcal{F}, Q)$ where for each $0 \leq t \leq T$,

$$W(t) = B(t) - \int_0^t F(s)ds.$$

Find the form of $F = (F(t), 0 \leq t \leq T)$ which is both necessary and sufficient for $\widetilde{Z}$ to be a Q-martingale. ***(3 marks)***

(v) Deduce that (under the equivalent probability measure Q for which $\widetilde{Z}$ is a martingale)

$$S(t) = se^{U(t)+(r-c)t},$$

where $U(t) \sim N\left(-\frac{1}{2}\sigma^2 t, \sigma^2 t\right)$, for all $0 \leq t \leq T$. ***(4 marks)***

(vi) The arbitrage price π at time zero for a European contingent claim X is

$$\pi = e^{-rT}\mathbb{E}_Q(X).$$

Express π in terms of the difference of two values of the cdf Φ for a standard normal in the case where X is the pay-off for a European put option having strike price k and terminal date T so that $X = \max\{k - S(T), 0\}$.

(6 marks)

6 (i) Suppose that the UK government issues a 25 year bond at 4% interest rate in 2011 with par value £100. Calculate a rational price that you might expect to be able to sell your bond for

(a) in 2013, when UK government 25 year bonds are then being issued at an interest rate of 6%.

(b) In 2015, when UK government 25 year bonds are then being issued at an interest rate of 2%.

How would you expect the market price of your bond to behave as you get closer to 2036? Briefly explain your answer.

(3 marks)

(ii) The term structure of a zero coupon bond that is worth £1 at the redemption time T is the family of random variables $(P(t,T), 0 \leq t \leq T, T \geq 0)$ defined on a probability space $(\Omega, \mathcal{F}, P)$ and we assume that each of these is differentiable with respect to T. The forward rate is defined to be

$$f(t,T) = -\frac{\partial}{\partial T}\log(P(t,T)),$$

and the yield is

$$y(t,T) = -\frac{1}{T-t}\log(P(t,T)).$$

Using differentiation and integration where necessary, express both $P(t,T)$ and $f(t,T)$ in terms of $y(t,T)$ and also show that the yield can be written in terms of the forward rate. ***(5 marks)***

(iii) The spot-rate is defined as $r(t) = f(t,t)$ and is assumed to be adapted to a given filtration $\{\mathcal{F}_t, 0 \leq t \leq T\}$. We use the spot-rate to construct an asset $(A(t), 0 \leq t \leq T)$ by defining

$$A(t) = \exp\left\{\int_0^t r(s)ds\right\}.$$

Assume that there exists a probability measure Q on $(\Omega, \mathcal{F})$ such that $(\tilde{P}(t,T), 0 \leq t \leq T)$ is a Q-martingale where

$$\tilde{P}(t,T) = A(t)^{-1}P(t,T),$$

for each $0 \leq t \leq T$. Deduce that

$$P(t,T) = \mathbb{E}_Q\left(\exp\left\{-\int_t^T r(s)ds\right\}\middle|\mathcal{F}_t\right).$$

(6 marks)

 Question 6 continued on next page

6 (continued)

(iv) The simplest model for the dynamical behaviour of the spot rate is the Vasicek model:

$$dr(t) = a(b - r(t))dt + \sigma dB(t),$$

where a, b and σ are positive constants and $(B(t), 0 \leq t \leq T)$ is Brownian motion. The unique solution to this stochastic differential equation is

$$r(t) = r(0)e^{-at} + b(1 - e^{-at}) + \sigma e^{-at} \int_0^t e^{as} dB(s).$$

Compute the mean and variance of $r(t)$ and write down its distribution. ***(6 marks)***

End of Question Paper