The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2010–2011

Fields - MAS438 — **2 hours 30 minutes**

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

1 (i) For each of the subsets J_1, J_2 of $\mathbb{C}$ specified below determine, with justification, whether it is a subfield of $\mathbb{C}$:

(a) $J_1 = \{a + b\sqrt{7} : \ a, b \in \mathbb{Q}\}$, ***(5 marks)***

(b) $J_2 = \{a + b\sqrt{7} + ci : \ a, b, c \in \mathbb{Q}\}$. ***(3 marks)***

(ii) Let K be a subfield of a field L. Give a definition of $[L : K]$. ***(2 marks)***

(iii) Consider the subfield $L = \mathbb{Q}(\sqrt{7}, \sqrt{5})$ of $\mathbb{C}$.

(a) Find $[L : \mathbb{Q}]$. Justify your answer and give a $\mathbb{Q}$-basis of L. ***(7 marks)***

(b) Prove that $L = \mathbb{Q}(\sqrt{5}, \sqrt{35})$. ***(3 marks)***

(c) Find $(1 + \sqrt{7} + \sqrt{5})^{-1}$. The answer should be given in terms of the basis of (a). ***(5 marks)***

2 (i) (a) Let $K \subseteq L$ be a field extension. Explain what it means to say that an element $a \in L$ is *algebraic over* K and what it means to say that L is *algebraic over* K. ***(2 marks)***

(b) Let $K \subseteq L$ be a finite field extension of degree n. Let $y \in L$. Show that the powers $y^0, y^1, y^2, \ldots, y^n$ are linearly dependent over K. Deduce that L is algebraic over K. ***(3 marks)***

(c) Let $K \subseteq L$ be a field extension. Explain what it means to say that an element $t \in L$ is *transcendental over* K. Suppose that $t \in L$ is a transcendental element over K. Find $[L : K]$. ***(4 marks)***

(ii) (a) Give an example of a primitive polynomial in $\mathbb{Z}[x]$ and a non-primitive polynomial in $\mathbb{Z}[x]$, both of degree 3. ***(2 marks)***

(b) Define the content $c(f)$ of a polynomial $f \in \mathbb{Z}[x]$. Is it true that $c(fg) = c(f)c(g)$? ***(2 marks)***

(iii) (a) State Eisenstein's Irreducibility Criterion. ***(2 marks)***

(b) Use a form of Eisenstein's Irreducibility Criterion to show that the following polynomials with integer coefficients are irreducible in $\mathbb{Q}[x]$:

$$-x^3 + 12x^2 - 6x + 2, \quad -1 + 12x - 6x^2 + 2x^3.$$

(3 marks)

(c) Prove Eisenstein's Irreducibility Criterion. You may assume without proof that if a non-constant polynomial $f \in \mathbb{Z}[x]$ is reducible in $\mathbb{Q}[x]$, then f can be written as a product of two non-constant polynomials in $\mathbb{Z}[x]$. ***(7 marks)***

3 (i) Let n be a positive integer.

(a) Give a definition of n-th *cyclotomic polynomial* $\phi_n(x)$. ***(2 marks)***

(b) Show that if p is a prime number then

$$\phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1.$$ ***(3 marks)***

(c) Find $\phi_n(x)$ for $n = 1, 2, 3, 4$. ***(3 marks)***

(d) Let p be a prime number. Prove that

$$\phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$$

is an irreducible polynomial in $\mathbb{Q}[x]$. ***(8 marks)***

(ii) Let $K \subseteq L$ be a field extension, $a \in L$ be an algebraic element over K and $m(x)$ be the minimal polynomial of the element a over K.

(a) Prove that $K(a) = \{\lambda_0 + \lambda_1 a + \cdots + \lambda_{n-1}a^{n-1} \mid \lambda_0, \ldots, \lambda_{n-1} \in K\}$ where $n = \deg(m(x))$. ***(6 marks)***

(b) Prove that the elements $1, a, a^2, \ldots, a^{n-1}$ are a K-basis for the vector space $K(a)$. ***(2 marks)***

(c) Find the degree $[K(a) : K]$. ***(1 mark)***

4 (i) (a) Specify the four standard constructions that are used in the theory of ruler-and-compass constructions and involve perpendicular and parallel lines. ***(5 marks)***

(b) Give the definition of a constructible real number. ***(2 marks)***

(c) Let $c \in \mathbb{R}$. Prove that c is a constructible real number if and only if $(0, c)$ is a constructible point in the plane. ***(4 marks)***

(d) Let $a, b \in \mathbb{R}$. Prove that (a, b) is a constructible point if and only if a and b are constructible real numbers. ***(4 marks)***

(e) Let $a, b \in \mathbb{R}$ be constructible numbers. Using Standard Constructions I–IV prove that the numbers

$$a - b \quad \text{and} \quad a + b$$

are constructible (You may use the fact that a point (x, y) is constructible if and only if the numbers x and y are constructible). ***(6 marks)***

(ii) Show that the number

$$\frac{\sqrt{\sqrt{2} + \sqrt[4]{3}}}{\sqrt{\sqrt{5} + \sqrt[8]{7}}}$$

is constructible. State clearly any result that you use. ***(4 marks)***

5 (i) Give the definition of a finite field extension. ***(2 marks)***

(ii) (a) State the Degrees Theorem. ***(3 marks)***

(b) Prove the Degrees Theorem. ***(8 marks)***

(iii) Give the definition of the splitting field for a polynomial $f(x) \in K[x]$ where K is a subfield of the field of complex numbers $\mathbb{C}$. ***(2 marks)***

(iv) (a) Let $z \in \mathbb{C}$ be a complex number. State (without proof) a necessary and sufficient condition on the field extension $\mathbb{Q} \subseteq \mathbb{Q}(z)$ for z to be constructible. ***(2 marks)***

(b) Apply the criterion from (a) to show that the number $\sqrt[4]{2}$ is constructible. ***(2 marks)***

(c) Explain what is meant by the problem of *squaring the circle*. ***(2 marks)***

(d) Explain why it is not possible to square the circle. ***(4 marks)***

End of Question Paper