The University Of Sheffield.

**SCHOOL OF MATHEMATICS AND STATISTICS** **Spring Semester 2010–2011**

**MAS422 Magnetohydrodynamics** **2 hours**

*Answer* ***four*** *questions. You are advised* ***not*** *to answer more than four questions: if you do, only your best four will be counted.*

**1** (i) Given $\mathbf{u} = (\sin z, \cos z, 1)$ and the initial magnetic field $\mathbf{B}(\mathbf{x}, 0) = (x, -y, 1)$, find $\mathbf{B}(\mathbf{x}, t)$ by obtaining the Lagrangian coordinates corresponding to $\mathbf{u}$ and applying the Cauchy solution. ***(15 marks)***

(ii) Write $\mathbf{B}(\mathbf{x}, t) = (x + g, -y + g, 1)$ obtained in (i) and verify by direct substitution that it is indeed the solution of the perfectly conducting induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B})$$

***(10 marks)***

**2** (i) Sketch the field lines for the following fields. Indicate clearly the direction of the field in each case.

(a) $\mathbf{B}$ = (1,1,0) ***(4 marks)***

(b) $\mathbf{B}$ = (-$y$,-$x$,0) ***(4 marks)***

(ii) For $B(x, t) = \phi(t)e^{-x^2/(4\eta t)}$ to satisfy

$$\frac{\partial B}{\partial t} = \eta \frac{\partial^2 B}{\partial x^2},$$

what is the condition on $\dfrac{d\phi}{dt}$? ***(6 marks)***

 **Question 2 continued on next page**

**2** (continued)

(iii) Consider the magnetic induction equation in the case where the magnetic diffusivity $\eta = 0$. Use the standard vector identity

$$\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F}) + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}$$

together with Maxwell's equation $\nabla \cdot \mathbf{B} = 0$ and the incompressibility condition $\nabla \cdot \mathbf{u} = 0$ to show that the induction equation can be rewritten as

$$\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{B} = (\mathbf{B} \cdot \nabla)\mathbf{u}$$

*(3 marks)*

(iv) Repeat (iii) for the case where the fluid is compressible, satisfying

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,$$

in place of $\nabla \cdot \mathbf{u}$=0. In this case, show that the induction equation may be written as

$$\frac{\partial}{\partial t}\left(\frac{\mathbf{B}}{\rho}\right) + (\mathbf{u} \cdot \nabla)\frac{\mathbf{B}}{\rho} = \left(\frac{\mathbf{B}}{\rho} \cdot \nabla\right)\mathbf{u}$$

*(8 marks)*

**3** (i) Magnetic field is given by

$$\mathbf{B} = (y, x, 0)$$

calculate

(a) $\mathbf{J} \times \mathbf{B}$ *(2 marks)*

(b) $(\mathbf{B} \cdot \nabla)\dfrac{\mathbf{B}}{\mu}$ *(2 marks)*

(c) $-\nabla\left(\dfrac{B^2}{2\mu}\right)$ *(2 marks)*

What do you expect the directions of the magnetic tension and pressure forces to be in the $x$-axis? Show them with the arrows. *(2 marks)*

 Question 3 continued on next page

**3** (continued)

(ii) The continuity equation can be written as

$$\frac{D\mathbf{u}}{Dt} = -\nabla p \ + \ \rho\mathbf{g}.$$

For $\mathbf{u} = 0$, write down the above equation for a region in cartesian coordinates $(x,\, y,\, z)$ such that everything is a function of $z$ alone and $\mathbf{g} = -g\hat{z}$ where $g$ is a constant.
For $p = K\rho^{1+\frac{1}{n}}$ with both $K$ and $n$ constant, then find $\rho$. ***(7 marks)***

(iii) If a magnetic field $\mathbf{B} = (B_R, B_\phi, B_z)$ varies with $R$ alone, why can it not possess a radial component $(B_R)$? ***(3 marks)***

(iv) Consider a coronal loop with $B_0 = 10$ Gauss, $L = 100$ Mm and $\rho_0 = 8 \times 10^{-13}$ kg m$^{-3}$ oscillating. Calculate

(a) the Alfveń speed ***(2 marks)***

(b) the wavenumber $k_z = \pi/L$ ***(2 marks)***

(c) the frequency $\omega = k_z v_A$ and the period of oscillation. ***(3 marks)***

[Hint: use $\mu_o = 4\pi \times 10^{-7}$ H/m and 1 Tesla $= 10^4$ Gauss.]

**4** (i) An inviscid, perfectly conducting, incompressible fluid, is permeated by a uniform magnetic field $\mathbf{B_0}$. The motion of the fluid is described by the momentum equation

$$\rho\left[\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U}\cdot\nabla)\mathbf{U}\right] = -\nabla p \ + \ \frac{(\nabla\times\mathbf{B})}{\mu_0}\times\mathbf{B}$$

and magnetic induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{U}\times\mathbf{B}).$$

The fluid is initially at rest (i.e. $\mathbf{U}_0 = 0$) and then given a small perturbation. Write down the linearised momentum and induction equations.

***(6 marks)***

(ii) Seeking solutions proportional to $\exp i(\mathbf{k}\cdot\mathbf{x} - \omega t)$ in the above linearised equations, show that

$$\omega^2 = \frac{(\mathbf{k}\cdot\mathbf{B_0})^2}{\mu_0\rho}$$

where $\mu_0$ is the magnetic permeability and $\rho$ the fluid density.

***(11 marks)***

(iii) The magnetic field $\mathbf{B}(x,t)$ in a plasma cell is given by

$$B\sin px = B_0 \sin px^*$$

where $\tan\frac{p}{2}x^* = \left(\tan\frac{p}{2}x\right)\mathrm{e}^{-pv_0t}$.

(a) Eliminate $x^*$ to find $B(x,t)$. [Hint: Use $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$]

***(6 marks)***

(b) What happens to $B$ in time at $(x = 0)$ and at $(x = \pi/p)$? ***(2 marks)***

**5** (i) A magnetic field is given by

$$\mathbf{B} = B_0 \left( \frac{y}{r^{2s}}, \frac{-x}{r^{2s}}, 0 \right),$$

where $r^2 = x^2 + y^2$, $B_0$ is a positive constant and $2s > 1$.

(a) Verify that $\nabla \cdot \mathbf{B} = 0$. ***(3 marks)***

(b) Sketch the field lines, indicating clearly the direction of the field. ***(6 marks)***

(c) Discuss the magnetic tension and pressure gradient forces acting on a parcel of fluid. Discuss how these forces depend on $s$ and $r$ and determine the value of $s$ for which they balance. [You may use without proof that the component of the magnetic tension force normal to $\mathbf{B}$ has strength $B^2/\mu_0 R_c$ where $R_c$ is the radius of curvature of the field line.] ***(6 marks)***

(d) Verify your conclusion about the balance of forces in (c) by explicitly calculating the Lorentz force. ***(5 marks)***

(ii) If the continuity equation is given by

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0$$

and the energy equation is given by

$$\frac{D}{Dt} \left( \frac{p}{\rho^\gamma} \right) = 0$$

where $\frac{D}{Dt}$ is the convective derivative i.e.

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla,$$

show that

$$\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p = -\gamma p \nabla \cdot \mathbf{v}$$

***(5 marks)***

## End of Question Paper