The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2010-2011**

Applicable Analysis **2 hours 30 minutes**

Answer ***four*** *questions. If you answer more than four questions, only your best four will be counted.*

You may use the following results when answering questions on this paper.

Table of Laplace Transforms	
Function	*Laplace Transform*
$t^{\alpha}e^{bt} (\alpha > -1)$	$\dfrac{\Gamma(\alpha+1)}{(s-b)^{\alpha+1}}$
$\sin at$	$\dfrac{a}{s^2+a^2}$
$\cos at$	$\dfrac{s}{s^2+a^2}$
$f(t)e^{bt}$	$F(s-b)$
$f^{(n)}(t)$	$s^nF(s) - \sum_{k=1}^{n} f^{(k-1)}(0)s^{n-k}$
$tf(t)$	$-F'(s)$

1 (i) Define what is meant by the statement that $\int_a^\infty f(x)\,dx$ exists.

(2 marks)

Prove, **from your definition,** each of the following statements:

(a) $\displaystyle\int_2^\infty \frac{2}{x^2-1}\,dx$ exists;

(b) $\displaystyle\int_2^\infty \frac{2x}{x^2-1}\,dx$ does not exist.

(5 marks)

(ii) State, without proof, the Comparison Test for convergence and divergence of integrals of the form $\int_a^\infty f(x)\,dx$. Your statement should include conditions under which the results are valid.

(4 marks)

Define what is meant by the statement that $\int_a^\infty f(x)dx$ is absolutely convergent.

(2 marks)

Prove each of the following, stating any standard results you need to use:

(a) $\displaystyle\int_2^\infty \frac{x(2+\cos x)}{x^2-1}\,dx$ diverges;

(b) $\displaystyle\int_2^\infty \frac{\sin x}{x^2-1}\,dx$ converges;

(c) $\displaystyle\int_0^\infty \frac{dx}{(x^5+1)^{1/4}}$ converges.

(12 marks)

2 (i) State, without proof, the theorem concerning differentiation of an integral of the form $\int_a^\infty f(x,y)\,dx$. Your statement should include conditions under which the result holds. ***(4 marks)***

Let

$$F(y) \quad = \quad \int_0^\infty e^{-x^2 y}\,dx \quad (y > 0)\,.$$

Prove that F is differentiable on every interval $[c, d]$ with $0 < c < d$. You may assume that $\int_0^\infty x^2 e^{-x^2}\,dx$ converges.

Show also that,

$$F'(y) \quad + \quad \frac{1}{2y}\,F(y) \quad = \quad 0 \qquad (*)$$

for $c \le y \le d$. ***(9 marks)***

Deduce that $(*)$ holds for all $y > 0$. ***(1 mark)***

By solving the differential equation $(*)$, find an expression for $F(y)$ in terms of y valid for $y > 0$. You may assume that $\int_0^\infty e^{-t^2} dt \quad = \quad \frac{1}{2}\sqrt{\pi}\,.$ ***(4 marks)***

(ii) Define the Γ function. ***(2 marks)***

Prove that

$$\int_0^\infty x^{13} e^{-x^4/4}\,dx \quad = \quad 60\sqrt{\pi}\,.$$ ***(5 marks)***

3 Define the Beta function. State, without proof, the relation between the Beta and Gamma functions. ***(3 marks)***

Prove that

$$B(x,y) \;=\; 2\int_0^{\pi/2} \cos^{2x-1}\theta \sin^{2y-1}\theta \, d\theta \qquad (x>0,\ y>0)$$

and

$$B(x,y) \;=\; \int_0^{\infty} \frac{u^{x-1}}{(1+u)^{x+y}}\, du \qquad (x>0,\ y>0).$$

(4 marks)

Prove each of the following, stating any standard results you need to use:

(i)

$$\int_0^{\frac{\pi}{2}} (\tan\theta)^{1/3}\, d\theta \;=\; \frac{\pi}{\sqrt{3}}\,;$$ ***(5 marks)***

(ii)

$$\int_{-\infty}^{\infty} \frac{e^{2x}}{(e^{3x}+1)^2}\, dx \;=\; \frac{2\pi}{9\sqrt{3}}\,.$$ ***(7 marks)***

(iii)

$$\int_0^1 \sqrt{x\,(1-x^3)}\, dx \;=\; \frac{\pi}{6}\,.$$ ***(6 marks)***

4 (i) Define what is meant by the statement that $\int_0^\infty e^{-st} f(t)\, dt$ has abscissa of convergence c. ***(1 mark)***

Verify that

$$t^2 + 1 \quad \geq \quad 2t \qquad (t \geq 0)\,.$$ ***(1 mark)***

Prove that $\displaystyle\int_0^\infty \frac{t\, e^{-st}}{t^2+1}\, dt$ has abscissa of convergence 0.

(5 marks)

(ii) In each of the following cases, find the function continuous on $[0, \infty)$, with the given Laplace transform:

(a) $\dfrac{1}{s(s+1)} \quad (s > 0)$;

(b) $\dfrac{s}{s^2 - 2s + 2} \quad (s > 1)$.

(5 marks)

(iii) Show that, for $s > 0$,

$$\int_0^\infty \frac{dx}{(x^2+1)(x^2+s^2)} \quad = \quad \frac{\pi}{2s(s+1)}\,.$$

By considering

$$\int_0^\infty \frac{\sin xt}{x\,(x^2+1)}\, dx$$

show that

$$\int_0^\infty \frac{\sin x}{x\,(x^2+1)}\, dx \quad = \quad \frac{\pi}{2}\,(1 - e^{-1})\,.$$

Find the value of

$$\int_0^\infty \frac{\sin x\ \cos x}{x(x^2+1)}\, dx\,.$$ ***(13 marks)***

5 (i) Let $f : [0, \infty) \to \mathbb{R}$ be continuous and suppose that the Laplace transform $F = L(f)$ exists on (c, ∞) for some $c \in \mathbb{R}$. State, without proof, the formula giving $L\left(\dfrac{f(t)}{t}\right)$ in terms of F. Your statement should include sufficient conditions to ensure the validity of the formula. ***(2 marks)***

Find the Laplace transform of each of the following functions

(a) $\sin^2 t\,,$

(b) $\dfrac{\sin^2 t}{t}\,.$

(10 marks)

(ii) Using Laplace transforms, solve the differential equation

$$t\,\frac{d^2y}{dt^2} + (1+2t)\,\frac{dy}{dt} + 2y \quad = \quad 4(1+t)\,e^t$$

subject to the initial conditions $y(0) = 1$ and $y'(0) = 2\,.$ ***(13 marks)***

End of Question Paper