The University Of Sheffield.

**SCHOOL OF MATHEMATICS AND STATISTICS** **Autumn Semester 2010–2011**

**Combinatorics** **2 hours 30 minutes**

*Answer* ***four*** *questions. If you answer more than four questions, only your best four will be counted.*

**1** (i) (a) State the Binomial Theorem. ***(2 marks)***

(b) By differentiating, deduce from the Binomial Theorem that

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1}.$$

***(4 marks)***

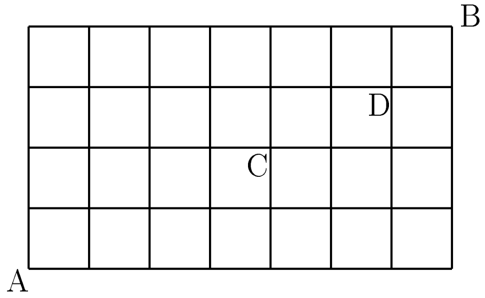
(ii) Let $n \geqslant 3$. Show, by each of the following two methods, that

$$\binom{n}{k} - \binom{n-3}{k} = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-3}{k-1}.$$

(a) Use Pascal's Identity. ***(4 marks)***

(b) Let $S$ be a set with $n$ elements, containing the elements $a$, $b$ and $c$. Consider the number of subsets of $S$ of size $k$ containing at least one of $a$, $b$ or $c$. ***(6 marks)***

(iii) This part of the question concerns routes in the grid illustrated:




(a) How many shortest routes are there from $A$ to $B$ along the lines of the grid? Give a brief reason for your answer. ***(3 marks)***

(b) Find the number of such routes which go through $C$. ***(2 marks)***

(c) Find the number of such routes which go through $C$ but not through $D$. ***(4 marks)***

**2** (i) Consider a prison consisting of 64 cells arranged like the squares of an 8-by-8 chessboard. There are doors between all adjoining cells. A prisoner in one of the corner cells is told that he will be released if he can get into the diagonally opposite corner cell after passing through every other cell exactly once. Can the prisoner do this? ***(4 marks)***

(ii) (a) State the Inclusion/Exclusion Principle. ***(3 marks)***

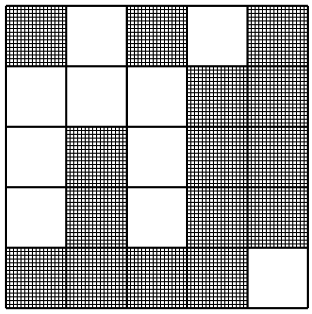
(b) Find the number of integers between 1 and $10,000$ that are neither perfect squares nor perfect cubes. ***(5 marks)***

(iii) (a) State the Pigeon-hole Principle. ***(2 marks)***

(b) Given 18 different positive even integers, show that there are two of them whose difference is divisible by 34. ***(5 marks)***

(c) Given 18 different positive even integers, all less than or equal to 500, show that there are two subsets of size 3 with the same sum. ***(6 marks)***

**3** (i) Calculate the rook polynomial of the (unshaded) board $B$:



***(8 marks)***

(ii) Let $B$ be part of an $n \times n$ board with rook polynomial

$$1 + r_1x + r_2x^2 + \cdots + r_nx^n$$

and let $\overline{B}$ be the complement of $B$. Prove that the number of ways of placing $n$ non-challenging rooks on $\overline{B}$ is

$$\sum_{k=0}^{n}(-1)^k(n-k)!r_k,$$

where $r_0 = 1$. ***(12 marks)***

(iii) Calculate the coefficient of $x^5$ in the rook polynomial of $\overline{B}$, where $B$ is the board in part (i). ***(2 marks)***

(iv) Draw a board which has rook polynomial $(1 + x)^2(1 + 4x + 2x^2)$. ***(3 marks)***

**4** (i) Define what it means for a matrix to be an $n \times n$ Latin square. ***(2 marks)***

(ii) For what value of $x$ can the following Latin rectangle be extended to a $6 \times 6$ Latin square?

$$\begin{pmatrix} 1 & x & 6 & 5 \\ 4 & 2 & 1 & 6 \\ 2 & 4 & 5 & 1 \end{pmatrix}$$

Write down one such extension. ***(8 marks)***

(iii) Let $A$ be the $7 \times 7$ matrix with entries $a_{ij}$, for $1 \leqslant i, j \leqslant 7$, where $a_{ij}$ is the element of $\{1, 2, \ldots, 7\}$ which is congruent to $2i + j - 2$ modulo 7.

(a) Show that $A = (a_{ij})$ is a $7 \times 7$ Latin square. ***(8 marks)***

(b) Define another $7 \times 7$ matrix $B$ to have entries $b_{ij}$, for $1 \leqslant i, j \leqslant 7$, where $b_{ij}$ is the element of $\{1, 2, \ldots, 7\}$ which is congruent to $3i+j-2$ modulo 7. You may assume that $B$ is a $7 \times 7$ Latin square. Show that $A$ and $B$ are orthogonal Latin squares. ***(7 marks)***

**5** (i) Which of the following are possible scores in a tournament of 8 people?

(a) $6, 6, 5, 5, 3, 2, 1, 1$.

(b) $6, 6, 6, 5, 2, 2, 1, 0$.

(c) $6, 6, 5, 5, 3, 2, 1, 0$. ***(8 marks)***

(ii) Consider a $4 \times 4$ board with squares labelled by the numbers $1, 2, \ldots, 16$ as shown.

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 5 | 6 | 7 | 8 |
| 9 | 10 | 11 | 12 |
| 13 | 14 | 15 | 16 |

Define blocks as follows. For each square on the board, form a block consisting of the six numbers appearing in the same row or the same column as that square (but not in the given square itself). For example, $\{2, 3, 4, 5, 9, 13\}$ is a block, corresponding to the top left square.

(a) Show that each number is in 6 blocks. ***(2 marks)***

(b) Show that each pair of numbers appears in precisely 2 blocks. ***(6 marks)***

(c) Deduce that the blocks make up a $(16, 16, 6, 6, 2)$ design. ***(3 marks)***

(d) By considering complement blocks, explain how to make a $(16, 16, 10, 10, 6)$ design. ***(6 marks)***

**End of Question Paper**