The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — **Autumn Semester 2010–11**

Topics in Number Theory (Level 3) — **2 hours 30 minutes**

Answer ***four*** *questions. If you answer more than four questions, only your best four will be counted.*

No credit will be given for solutions which rely solely on the use of a calculator. Your solutions should give enough details to make it clear how you arrived at your answers.

1 (i) You publish $(n, e) = (133, 13)$ in the RSA directory and receive 15. Decode it. ***(12 marks)***

(ii) State *Wilson's Theorem* and determine the remainder when 26! is divided by

(a) 27,

(b) 29,

(c) 87,

(d) 2^{24}. ***(13 marks)***

2 (i) State the *Law of Quadratic Reciprocity.* ***(2 marks)***

(ii) Calculate the Legendre symbol $\left(\frac{76}{103}\right)$ and deduce whether the congruence

$$x^2 + 12x + 17 \equiv 0 \pmod{103}$$

has a solution. If it has, solve it. ***(9 marks)***

(iii) Use the Law of Quadratic Reciprocity to prove that, for a prime number $p > 3$,

$$\left(\frac{3}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \text{ or } 11 \pmod{12} \\ -1 & \text{if } p \equiv 5 \text{ or } 7 \pmod{12}. \end{cases}$$

(8 marks)

Deduce that, if $p > 3$ is a prime divisor of $m^2 - 3n^2$, where m, n are coprime positive integers, then $p \equiv 1$ or $11 \pmod{12}$. Illustrate this in the case $(m, n) = (17, 3)$. ***(6 marks)***

3 (i) State *Euler's Criterion.*
For each of the numbers

$$2^{81}-1, \quad 2^{82}+1, \quad 2^{83}-1, \quad 2^{84}+1,$$

find a prime number which divides it. ***(10 marks)***

(ii) Show that, for the numbers 1184 and 1210, the sum of the proper positive divisors of each one is equal to the other. ***(10 marks)***

(iii) Define a *perfect number* and show that a power of a prime number cannot be perfect. ***(5 marks)***

4 (i) Determine all primitive Pythagorean triples in which one of the numbers is 208. ***(6 marks)***

(ii) Give two non-primitive Pythagorean triples in which one of the numbers is 208. ***(2 marks)***

(iii) Determine all primitive Pythagorean triples in which one of the numbers is 305. ***(11 marks)***

(iv) Give four non-primitive Pythagorean triples in which one of the numbers is 305. ***(6 marks)***

5 (i) Express $\sqrt{2}$ as a continued fraction and find a convergent of $\sqrt{2}$ which differs from it by less than 10^{-4}. Find three solutions of the Pell equation $x^2-2y^2=1$ in positive integers, and hence find three primitive Pythagorean triples x, y, z with x even such that $y-x=1$. ***(17 marks)***

(ii) Express the continued fraction $[1;\ 2,\ \overline{3}\ ]$ in the form $a+b\sqrt{c}$, where a, b are rational numbers and c is a positive integer. ***(8 marks)***

End of Question Paper