The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2010-2011**

Milestones in Applied Mathematics II **2 Hours**

Marks will be awared for your best FOUR answers.

1 (i) A radio transmitter of power 10 kW emits waves of wavelength 100 m. You are given that the speed of the waves is 3×10^8 m s^{-1}, and Planck's constant $h = 6.626 \times 10^{-34}$J s. Calculate, correct to three significant figures

(a) The number of photons emitted per second.

(b) The momentum of each photon.

(c) The number of photons per second hitting an aerial of area 1 m^2 a distance of 10 km from the transmitter. ***(13 marks)***

(ii) Calculate the energy of the quantum state represented by the wavefunction

$$\psi(x, t) = \exp\left(\frac{mx^2}{2\hbar}\right)\exp\left(\frac{it}{2}\right),$$

and find the potential $V(x)$ for which this wave-function is a solution of the time-dependent Schrödinger equation.

Is ψ a realistic possible wavefunction? Give a reason for your answer. ***(12 marks)***

2 (i) A quantum mechanical particle of mass m moves freely in the interval $[0, a]$ on the x-axis. Assuming that the energy is positive, show that the energy levels are

$$E_n = \frac{n^2\hbar^2\pi^2}{2ma^2} \qquad n = 1, 2, 3, \ldots,$$

and that the corresponding solutions of the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} + V(x)\Psi = E\Psi,$$

have the form

$$\Psi(x) = \begin{cases} A_n \sin\left(\dfrac{n\pi x}{a}\right), & 0 < x < a; \\ 0, & \text{otherwise.} \end{cases}$$

where the A_n are *undetermined* constants. ***(12 marks)***

(ii) A quantum mechanical particle of mass m moves freely in the rectangle $0 < x < a$, $0 < y < 2a$. Use your answer to part (i) above to show that the energy levels of this system are

$$E = \frac{\hbar^2\pi^2}{8ma^2}\left(4p^2 + q^2\right),$$

where $p, q = 1, 2, 3\ldots$.

Show that the state with energy

$$E = \frac{5\hbar^2\pi^2}{2ma^2}$$

is doubly-degenerate. ***(13 marks)***

3 A quantum mechanical particle of mass m moves in the potential

$$V(x) = \begin{cases} 0 & 0 < x < a; \\ V_0 & a < x < b; \\ \infty & \text{otherwise,} \end{cases}$$

where V_0 is a positive real constant.

The energy E of the particle is such that $E > V_0$.

Define positive real constants k and l by

$$E = \frac{\hbar^2 k^2}{2m}, \qquad E - V_0 = \frac{\hbar^2 l^2}{2m}.$$

Show that the time-independent Schrödinger equation takes the form

$$\frac{d^2\Psi}{dx^2} + k^2\Psi = 0 \qquad 0 < x < a;$$
$$\frac{d^2\Psi}{dx^2} + l^2\Psi = 0 \qquad a < x < b;$$

and state the boundary conditions that should be satisfied by the wave-function $\Psi(x)$ at $x = 0, a, b$. ***(10 marks)***

Hence show that solutions of the Schrödinger equation satisfying the boundary conditions are possible provided that

$$\frac{1}{k}\tan(ka) = \frac{1}{l}\tan l(a - b).$$

(15 marks)

4 The Hamiltonian of a quantum system is given by

$$H = \hbar\omega \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Find the energy eigenvalues of the system and the corresponding normalized eigenvectors. ***(10 marks)***

At time $t = 0$, the state of the system is given by

$$\Psi(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Show that, at a later time t, the state of the system is given by

$$\Psi(t) = \frac{1}{2}e^{-i\omega t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}e^{i\omega t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Show that the probability that, at time t, the system is observed to be in the state

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

is $\sin^2(\omega t)$. ***(15 marks)***

5 Show that the commutator of the position operator X and momentum operator P is given by

$$[X, P] = i\hbar.$$

(7 marks)

Define operators A and B by

$$A = P + iX, \qquad\qquad B = P - iX.$$

Show that

$$[A, B] = -2\hbar.$$

(5 marks)

The Hamiltonian of a quantum system is given by

$$H = \frac{1}{2}P^2 + \frac{1}{2}X^2.$$

Show that $[H, A] = \hbar A$, and hence that

$$HA = AH + \hbar A. \qquad\qquad (*)$$

(7 marks)

Suppose that ψ is an eigenvector of H with eigenvalue E. By applying the operator $(*)$ to ψ, show that $A\psi$ is an eigenvector of H and determine the eigenvalue.

(6 marks)

End of Question Paper