The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS Spring Semester 2010–2011

OPERATIONS RESEARCH 2 hours

Answer ***four*** *questions. You are advised* ***not*** *to answer more than four questions: if you do, only your best four will be counted.*

1 (i) A company owns a steam turbine power generating plant that generates its steam from coal. It wishes to maximise the steam output. This, however, may result in emission that does not meet the Environmental Protection Agency standards. The regulations limit sulphur dioxide discharge to 2000 parts per million in the smoke, and smoke discharge from the plant to 20 kg per hour. The plant receives two grades of pulverised coal, C1 and C2, for use in the steam plant.

The two grades are usually mixed together before burning. It can be assumed that the amount of sulphur pollutant discharged (in parts per million) is a weighted average of each of the two coal grades used in the mixture. The following data are based upon the consumption of 1000 kg per hour of each of the two coal grades.

coal grade	sulphur discharge in parts per million	smoke discharge in kg per hour	steam generated in kg per hour
C1	1800	2.1	12000
C2	2100	0.9	9000

Formulate the linear programming model to determine the optimal amounts of types C1 and C2 coal burnt. **Do not solve the problem.** ***(6 marks)***

1 (continued)

(ii) Use the Two-Phase Simplex method to solve the problem

$$\min \quad z = 2x_1 - 3x_2$$

subject to

$$\begin{aligned} 0.5x_1 + 0.25x_2 &\leq 4 \\ x_1 + 3x_2 &\geq 20 \\ x_1 + x_2 &= 10 \\ x_1, x_2 &\geq 0 \ . \end{aligned}$$

Note that Phase 1 should be completed within three tableaux.

(15 marks)

(iii) Use the Steepest Descent algorithm to establish the next search direction to determine the approximate minimum point of the function

$$f(x, y) = x^2 + y^2 + xysin(xy)$$

from the point $x = \pi/3$, $y = 1$. ***(4 marks)***

2 (i) From **first principles**, derive the dual of the linear programming problem

$$\min \quad v = y_1 - y_2$$

subject to

$$\begin{aligned} 3y_1 + y_2 &= 6 \\ y_1 + 2y_2 &\geq 3 \\ y_1 + y_2 &\leq 8 \\ y_1, y_2 &\geq 0. \end{aligned}$$

(8 marks)

(ii) Use the **canonical** form of the following problem to derive its dual

$$\max \quad z = 6x_1 + x_2 - 2x_3$$

subject to

$$\begin{aligned} 3x_1 + 2x_2 - x_3 &\leq 6 \\ x_1 - x_2 + 2x_3 &\geq 5 \\ x_2 + x_3 &= 10 \\ x_1 \text{ unrestricted ;} \quad x_2, x_3 &\geq 0 \end{aligned}$$

(7 marks)

(iii) The optimal tableau for the problem

$$\max \quad z = x_1 + 2x_2 + x_3$$

subject to

$$\begin{aligned} x_1 + x_2 + x_3 &\leq 12 \\ 2x_1 + x_2 - x_3 &\leq 6 \\ x_1 - x_2 + 2x_3 &\leq 8 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

is

Basic	z	x_1	x_2	x_3	x_4	x_5	x_6	Soln
z−row	1	3/2	0	0	3/2	1/2	0	21
x_3	1	−1/2	0	1	1/2	−1/2	0	3
x_2	2	3/2	1	0	1/2	1/2	0	9
x_6	0	7/2	0	0	−1/2	3/2	1	11

By adding an additional row to the tableau, verify that the addition of the constraint

$$x_1 + 2x_2 - x_3 \leq 12$$

changes the optimal solution. Determine the new optimal solution.

(10 marks)

3 (i) Consider the problem

$$\max \quad z = \mathbf{c}_1^T \mathbf{x}_1 + \mathbf{c}_2^T \mathbf{x}_2$$

subject to

$$\begin{aligned} A\mathbf{x}_1 + \mathbf{x}_2 &= \mathbf{b} \\ \mathbf{x}_1, \mathbf{x}_2 &\geq \mathbf{0} \end{aligned}$$

The tableau, at a stage in the Simplex algorithm when the basis matrix is B, is represented by

$$\begin{bmatrix} 1 & \mathbf{c}_B^T B^{-1} A - \mathbf{c}_1^T & \mathbf{c}_B^T B^{-1} - \mathbf{c}_2^T \\ \mathbf{0} & B^{-1} A & B^{-1} \end{bmatrix} \begin{bmatrix} z \\ \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{c}_B^T B^{-1} \mathbf{b} \\ B^{-1} \mathbf{b} \end{bmatrix}$$

where $\mathbf{x}_2$ is the starting basis.

Write down the necesssary conditions for this tableau to be optimal. Write down an expression for the optimal dual variables in terms of the notation above and verify that the dual constraints are satisfied.

(10 marks)

(ii) Verify, by using the Two-Phase Simplex method, that the following problem has no feasible solution:

$$\min \quad z = x_1 + x_2$$

subject to

$$\begin{aligned} -x_1 + x_2 &\geq 3 \\ x_1 + x_2 &\leq 2 \\ x_1,\ x_2 &\geq 0 \end{aligned}$$

Do not simply show that the inequality constraints are inconsistent.

(10 marks)

(iii) For each of the following sub-problems of a different Linear Programming problem set up the appropriate 0-1 integer programming sub-problem.

(a) If $x_1 + 2x_2 + x_3 > 9$, then $x_7 + 4x_8 \geq 6$.

(b) Only two of the constraints $x_1 + x_2 \leq 7$, $x_2 - x_3 \leq 5$ and $x_3 \leq 3$ are required to hold at the optimal solution.

(5 marks)

4 (i) A zero-sum two-player game has player A's payoff matrix

	B_1	B_2	B_3
A_1	2	4	5
A_2	10	7	q
A_3	4	p	6

Player A wishes to maximise his/her payoff. Determine the values of p (row player strategy) and q (column player strategy) that will make the entry (2,2) a saddle point.

(5 marks)

(ii) Consider a zero-sum game with the row player's payoff matrix having all its elements positive. Show that the optimal solution of the row player's problem (p_i) satisfies a primal maximising Linear Programming problem and that the optimal solution of the column player's problem (q_j) satisfies the dual minimising Linear Programming problem. ***(6 marks)***

For the following game, set up the two Linear Programming problems that can be solved to obtain the optimal mixed strategies.

Do not remove any variables by considering domination.

	B_1	B_2	B_3
A_1	3	-1	-3
A_2	-2	4	-1

Use a graphical technique to solve the two-dimensional row player LP problem. Determine the optimal solution for the row player and the value of the game.

For the column player problem, explain, by studying the row player LP solution, why the optimal value of one of the strategies is zero.

(14 marks)

5 A factory manufactures two types of flour.

Unit weight of flour 1 can be sold at £25, requires one hour's work from worker 1, two hours' work from worker 2 and is made from raw material worth £5.

Unit weight of flour 2 can be sold at £22, requires two hours' work from worker 1, one hour's work from worker 2 and is made from raw material worth £4.

Worker 1 can work up to 40 hours per week and is paid £5 per hour. Worker 2 can work up to 50 hours per week and is paid £6 per hour.

There is an unlimited supply of raw material.

Let x_i be the amount of type i flour produced and S_j the slack variable relating to the working hours of worker j. One can formulate the Linear Programming problem:

$$\max \quad z = 3x_1 + 2x_2 \ .$$

subject to

$$\begin{aligned} x_1 + 2x_2 &\leq 40 \\ 2x_1 + x_2 &\leq 50 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Do not derive the above formulation.

The optimal tableau is

	x_1	x_2	S_1	S_2	Soln
z	0	0	1/3	4/3	80
x_1	1	0	−1/3	2/3	20
x_2	0	1	2/3	−1/3	10

(i) Write down the optimal solution. ***(3 marks)***

(ii) Determine the possible range of prices of type 1 flour for which the current basis would remain optimal. ***(4 marks)***

(iii) If worker 1 were willing to work only 30 hours per week, would the current basic solution remain feasible? ***(2 marks)***

(iv) If worker 2 were willing to work up to 60 hours per week, would the current basic solution remain optimal? ***(2 marks)***

 Question 5 continued on next page

5 (continued)

(v) If worker 1 were willing to work an additional hour, what is the most that the factory should be willing to pay? ***(4 marks)***

(vi) If worker 2 were willing to work only 48 hours per week, determine the new profit value and the amount of flour of each type that would be produced. ***(5 marks)***

(vii) Type 3 flour is under consideration for production. Its specifications are: price £20, 2 hours' work from worker 1, 2 hours' work from worker 2, cost of raw materials £2. Determine whether the factory should manufacture type 3 flour. ***(5 marks)***

End of Question Paper