The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2010–2011

Fluid Mechanics I — **2 hours**

Answer ***four*** *questions. You are advised* ***not*** *to answer more than four questions: if you do, only your best four will be counted.*

1 (i) Consider the following two flows:

(a)

$$\mathbf{u} = (-\Omega y, \Omega x, 0),$$

(b)

$$\mathbf{u} = \left(-\frac{\Omega y}{x^2+y^2}, \frac{\Omega x}{x^2+y^2}, 0\right) \quad \text{for } x^2+y^2 \neq 0,$$

where Ω is constant. Compute the vorticity in each case and state which flow is rotational and which is irrotational. ***(6 marks)***

(ii) For the above flows, draw stream lines for both (a) and (b). ***(4 marks)***

(iii) The material derivative of vector $\mathbf{A}$ is defined as

$$\frac{D\mathbf{A}}{Dt} = \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{A},$$

where $\mathbf{u}$ denotes fluid velocity which is in general compressible.

Deduce the following formula:

$$\frac{D\mathbf{A}}{Dt} = \frac{\partial \mathbf{A}}{\partial t} + \frac{1}{2}\left[\nabla(\mathbf{u}\cdot\mathbf{A}) + (\nabla\times\mathbf{u})\times\mathbf{A}\right.$$

$$\left.+(\nabla\times\mathbf{A})\times\mathbf{u} - \nabla\times(\mathbf{u}\times\mathbf{A}) + \mathbf{u}\nabla\cdot\mathbf{A} - \mathbf{A}\nabla\cdot\mathbf{u}\right]$$

You may use the following vector identities:

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}),$$

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = -(\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}).$$

(10 marks)

(iv) Using the result in (iii), derive the acceleration formula

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + \nabla\left(\frac{|\mathbf{u}|^2}{2}\right) + (\nabla\times\mathbf{u})\times\mathbf{u}.$$

(5 marks)

2 Consider viscous incompressible fluid on a half plane: $0 < y < \infty$, bounded by a solid plate at $y = 0$. The fluid is at rest at $t < 0$ and the plate is jerked into motion at $t = 0$ and thereafter moving with a constant velocity $(U, 0)$. Hence the initial condition for the x-component of velocity $u(y, t)$ is

$$u(y, 0) = 0 \text{ for } y > 0$$

and the boundary conditions are

$$u(0, t) = U \text{ for } t > 0,$$

$$u(\infty, t) = 0 \text{ for } t > 0$$

We assume that the flow is two-dimensional $\boldsymbol{v} = (u, v)$ with no pressure gradient and that no quantity depends on x.

(i) By using the equation of continuity, show that the y-component of velocity vanishes, that is, $v = 0$ everywhere. ***(3 marks)***

(ii) Show that $u(y, t)$ satisfies the equation

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2},$$

where ν is the kinematic viscosity. ***(5 marks)***

(iii) By introducing the new variables

$$u = Uf(\eta), \ \eta = \frac{y}{\sqrt{4\nu t}}$$

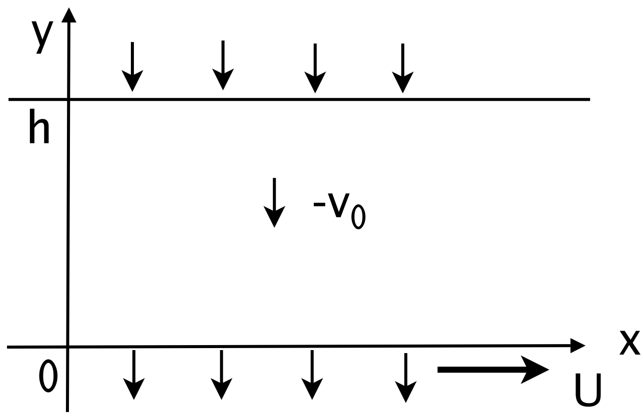
show that the above equation can be written as

$$\frac{d^2 f}{d\eta^2} + 2\eta \frac{df}{d\eta} = 0.$$

(9 marks)

(iv) State the boundary conditions for $f(\eta)$. ***(2 marks)***

(v) Solve the above equation for $f(\eta)$.
You may use the error function $\text{erf}(\eta) = \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-s^2} ds$, or its equivalent. ***(6 marks)***



3 Consider a viscous incompressible fluid of constant density ρ and of the kinematic viscosity ν, between two rigid boundaries $y = 0$ and $y = h$. The lower boundary moves in the x-direction with a constant speed U and the upper one is held fixed at rest. The boundaries are assumed to be *porous*, that is, they have tiny holes through which fluid can be sucked in and out.

The vertical velocity of the fluid is given by $v = -v_0$(constant) at all time. The problem is two-dimensional, that is, no quantities depend on z.

(i) Assuming that the horizontal velocity u and the pressure p are functions only of y, write down the steady (that is, time independent) Navier-Stokes equations for the problem.

(5 marks)

(ii) Deduce that the pressure p is constant, that is, independent of the spatial coordinates.

(2 marks)

(iii) Solve the equation for u to show that

$$u = U\frac{\exp\left(-\frac{v_0 y}{\nu}\right) - \exp\left(-\frac{v_0 h}{\nu}\right)}{1 - \exp\left(-\frac{v_0 h}{\nu}\right)}.$$

(10 marks)

(iv) Deduce an approximate formula for velocity which is valid close to the lower boundary, when $v_o h/\nu$ is large in comparison to 1. Sketch the velocity profile therein.

(5 marks)

(v) From the expression u in part (iii), derive the velocity u for the limiting case of $v_0 \to 0$.

(3 marks)

4 Consider a two-dimensional, incompressible viscous fluid in the region between two infinite horizontal plates along $y = 0$ and $y = h$, flowing steadily in the x-direction.

(i) Assuming that $\boldsymbol{v} = (u(x, y), 0)$ show that u is actually a function of y only. ***(2 marks)***

(ii) Write down the components of the Navier-Stokes equations by neglecting external body forces and show that the pressure gradient is constant and acts only in the x-direction. ***(12 marks)***

(iii) If the plates are fixed and $\dfrac{dp}{dx} = -P$, show that

$$\boldsymbol{v} = \frac{-P}{2\mu} y(y - h)\boldsymbol{e}_1,$$

where $\boldsymbol{e}_1$ is a unit vector in the x-direction and μ denotes the viscosity. ***(5 marks)***

(iv) Calculate the volume flux per unit area across any fixed plane perpendicular to the motion. ***(3 marks)***

(v) Calculate the drag per unit area exerted by the fluid on the plane $y = 0$. ***(3 marks)***

5 (i) A fluid moves in a steady two-dimensional flow in the region defined by $x \geq 0, \ y \geq 0$. The boundary with equation $y = 0$ is occupied by a stationary flat plate. Given that the x-component of velocity $u \to U$ as $y \to \infty$, where U is a constant, write down the expressions for
(a) the displacement thickness δ_1 of the boundary layer,
and
(b) the momentum thickness δ_2 of the boundary layer.

(4 marks)

(c) When the flow is given approximately by

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^m \quad \text{where } 0 < m < 1,$$

compute δ_1 and δ_2 explicitly and state which is the largest.

(6 marks)

(ii) Consider a steady two-dimensional flow past a semi-infinite solid boundary along $y = 0$ in the region $y \geq 0, \ x \geq 0$. Blasius's boundary layer equations for it are given by

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2},$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

where ν is the kinematic viscosity. Derive the energy equation for the boundary layer

$$\frac{d}{dx}\int_0^\infty \frac{u}{U}\left(1 - \frac{u^2}{U^2}\right) dy = \frac{2D}{\rho U^3},$$

where $u, \ v$ are the x- and y- components of the velocity and

$$D = \mu\int_0^\infty \left(\frac{\partial u}{\partial y}\right)^2 dy$$

is the dissipation rate of energy. The constant U is the x-component of the velocity as $y \to \infty$. ***(15 marks)***

End of Question Paper