The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2010–11**

MAS314 **2 hours**

Marks will be awarded for your best ***FOUR*** *answers.*

1 The inertial frame $\tilde{R} : (c\tilde{t}, \tilde{x})$ has constant, non-zero, velocity v in the x-direction relative to the inertial frame $R : (ct, x)$. The inertial frames $R : (ct, x)$ and $\tilde{R} : (c\tilde{t}, \tilde{x})$ are related by the homogeneous Lorentz transformation

$$\begin{pmatrix} c\tilde{t} \\ \tilde{x} \end{pmatrix} = \gamma(v) \begin{pmatrix} 1 & -\dfrac{v}{c} \\ -\dfrac{v}{c} & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix},$$

where

$$\gamma(v) = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}},$$

and c is the speed of light.

In the frame R, two points A and B are a distance D apart. You may assume that the point A is such that $x = 0$ and the point B is such that $x = D$ in the frame R.

An observer is travelling with uniform velocity v in the direction $\overrightarrow{AB}$ relative to the frame R. At time $t = 0$ in the frame R, the observer is at the point A.

At time $t = 0$, a firework explodes at each of the points A and B.

Find the co-ordinates of the corresponding events in the frame $\tilde{R}$, and deduce that the fireworks do *not* explode simultaneously in the frame $\tilde{R}$. ***(14 marks)***

A third firework explodes at time $t = T$ in the frame R, at the point C which is a distance $x = k$ from A in the frame R.

Find the co-ordinates of the corresponding event in the frame $\tilde{R}$. ***(5 marks)***

Show that it is not possible for the observer to observe the explosions at A and C to be simultaneous, unless $k > cT$. ***(6 marks)***

2 Two inertial frames $R : (ct, x, y, z)$ and $\tilde{R} : (c\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z})$ are related by the transformation

$$\begin{pmatrix} c\tilde{t} \\ \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = L \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}, \qquad (*)$$

where L is a (4×4)-matrix with constant entries.

Write down the conditions on the matrix L for the transformation $(*)$ to be a *proper orthochronous Lorentz transformation.* ***(3 marks)***

Consider the transformation $(*)$ with the matrix L given by

$$L = \begin{pmatrix} \sqrt{3} & \sqrt{2} & 0 & 0 \\ \sqrt{2} & \sqrt{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \qquad (**)$$

Show that the transformation $(*)$ is a proper orthochronous Lorentz transformation in this case. ***(11 marks)***

In the inertial frame R, a particle is moving with three-velocity, as measured in R, given by $(v_1, v_2, 0)$. Its three-velocity as measured in the inertial frame $\tilde{R}$ is $(w_1, w_2, 0)$.

Use the Lorentz transformation $(*)$ with L given by $(**)$ to find the components w_1 and w_2 in terms of v_1 and v_2. ***(11 marks)***

3 Let X and Y be any two four-vectors and let $Z = X + Y$.

Show that
$$g(Z, Z) = g(X, X) + 2g(X, Y) + g(Y, Y).$$

Deduce that, if X and Y are both null, then
$$g(Z, Z) = 2g(X, Y).$$

(4 marks)

For the rest of this question, assume that X and Y are future-pointing null four-vectors.

Show that Z is future-pointing. Now suppose that there is an inertial frame R in which X has components
$$X = (1, 1, 0, 0),$$
and Y has components
$$Y = (\alpha, \beta, \gamma, \delta)\,.$$

Show that, in the inertial frame R,
$$g(X, Y) = \alpha - \beta.$$

Use the fact that Y is a null four-vector to show that
$$\alpha^2 - \beta^2 \geq 0.$$

(10 marks)

Use this result to show that $Z = X + Y$ is either timelike or null. Find examples of future-pointing null four-vectors X and Y such that $X + Y$ is (a) timelike, and (b) null. ***(11 marks)***

4 The world-line of an observer in an inertial frame R is given by

$$X(\tau) = \frac{c^2}{a}\left(\sinh\left[\frac{a\tau}{c}\right], \cosh\left[\frac{a\tau}{c}\right]\right),$$

where a is a constant and c is the speed of light.

(i) Show that the velocity of the observer in the frame R is

$$v = c\tanh\left[\frac{a\tau}{c}\right].$$

(8 marks)

(ii) Show that

$$\gamma(v) = \cosh\left[\frac{a\tau}{c}\right].$$

And, hence show that τ is the proper time of the observer. ***(11 marks)***

(iii) Find the four-velocity V of the observer. If A is the four-acceleration of the observer, show that

$$A = \frac{a^2}{c^2}X.$$

(6 marks)

5 A particle of rest mass m has three-velocity $\boldsymbol{v}$ in an inertial frame R.

Write down expressions for the *energy* E, *three-momentum* $\boldsymbol{p}$, and *four-momentum* P of the particle in the inertial frame R. ***(4 marks)***

Two protons, each of mass m, are moving with speed v in opposite directions along the x-axis of an inertial frame R. The two protons collide and two new particles are formed, each with rest mass $10m$.

Assuming the new particles are formed at rest in R, show that $\gamma(v) = 10$ and hence find the energy of each proton before the collision in terms of m and c only. ***(6 marks)***

Suppose now that the protons each have the same energy before they collide. However, before the collision one proton is moving along the x-axis, while the other is moving along the y-axis in R. On collision the protons form a single new particle. For the remainder of this question, you may use the value $\gamma(v) = 10$ calculated above.

Show that the rest mass of the new particle is $\sqrt{202}m$ and find, in terms of m and c only, its energy and the magnitude of its three-momentum in the frame R. ***(15 marks)***

End of Question Paper