The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring semester 2010-2011**

Applied Differential Equations **2 hours**

Attempt all FOUR questions.

1 (i) The Euler 2 method, applied to the equation $dy/dx = f(x, y)$, is defined as:

$$k_1 = hf(x_n, y_n) \ ,$$
$$k_2 = hf(x_n + h, y_n + k_1) \ ,$$
$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2) \ .$$

Given the ordinary differential equation with the initial condition:

$$\frac{dy}{dx} = 2x + y^2 - 2; \quad y(0) = 1, \tag{1}$$

use the Euler 2 method to find the approximate solution at $x = 0.2$ using step-size $h = 0.1$. Work throughout correct to four decimal places. ***(9 marks)***

(ii) The Euler 2 method is used to solve equation (1) as far as $x = 2$, with two different step sizes. The results are

h	$y(2)$
0.05	2.86233
0.2	2.66393

Use these data to estimate the step size, h, required to ensure that the absolute value of the global discretization error in $y(2)$ is smaller than 5×10^{-3}. You may assume that the Euler 2 is a second order method. ***(7 marks)***

(iii) Apply the Euler 2 method to the test equation $y'(x) = \lambda y(x)$, where λ is a constant, hence show that $y_{n+1} = R(\overline{h})y_n$, where $\overline{h} = \lambda h$ and $R(\overline{h})$ is a second order polynomial in $\overline{h}$. Find the expression for $R(\overline{h})$, hence show that the interval of absolute stability for the Euler 2 method is $(-2, 0)$. ***(9 marks)***

2 (i) Determine the convergence, or the lack of it, of the linear multi-step method:

$$y_{n+1} = y_n + \frac{h}{12}(23f_n - 16f_{n-1} + 5f_{n-2}),$$

and the single-step method:

$$k_1 = hf_n, \quad k_2 = hf\left(x_n + \frac{3}{2}h, y_n + \frac{3}{2}k_1\right),$$
$$y_{n+1} = y_n + \frac{2}{3}k_1 + \frac{1}{3}k_2\ ,$$

where, in both cases, $f_n = f(x_n, y_n)$. State your arguments. ***(19 marks)***

(ii) The following table contains the grid-point values of the two solutions, $Y_1(x)$ and $Y_2(x)$, of a linear differential equation $d^2y/dx^2 = f(x, y, y')$ obtained using the fourth-order Runge-Kutta method.

x	2.0	3.0	4.0
$Y_1(x)$	1.36831	2.12856	3.41281
$Y_2(x)$	2.15473	3.35557	5.85697

$Y_1(x)$ was determined using the initial conditions $y(1) = 1$, $y'(1) = 0$, and $Y_2(x)$ was obtained using $y(1) = 1$, $y'(1) = 0.5$. Form a linear combination of these two solutions which is the numerical solution to the equation $d^2y/dx^2 = f(x, y, y')$ with the boundary conditions

$$y(1) = 1, \quad y(4) = 2.$$

Calculate the value of this solution at each x-value given in the table. ***(6 marks)***

3 (i) The function $u(x,t)$ satisfies the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad (0 \le x \le \ell, t \ge 0), \tag{2}$$

subject to the boundary conditions

$$u(0,t) = 0, \quad u(\ell,t) = 0,$$

and the initial condition

$$u(x,0) = 2\sin(\pi x/\ell) + \sin(2\pi x/\ell).$$

For a separable solution of the form $T(t)X(x)$, show that $X(x)$ satisfies the differential equation and the associated boundary condition:

$$X''(x) - \alpha X(x) = 0, \quad X(0) = X(\ell) = 0,$$

where α is a constant. Show that $T(t)$ satisfies the equation $T'(t) = k\alpha T(t)$. ***(6 marks)***

(ii) Assuming $\alpha = -s^2$ $(s > 0)$, find the values of s such that there are non-trivial solutions for $X(x)$ and $T(t)$, and thus find $X(x)$ and $T(t)$. ***(7 marks)***

(iii) The general solution of equation (2), with the given boundary conditions, can be written as

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell} \exp\left(\frac{-n^2\pi^2 kt}{\ell^2}\right),$$

where b_n are constants to be determined. Find the values of b_n such that $u(x,t)$ satisfies the given initial condition, hence write down the solution for $u(x,t)$. ***(4 marks)***

(iv) The function $w(x,t)$ satisfies the inhomogeneous heat equation

$$\frac{\partial w}{\partial t} = k\frac{\partial^2 w}{\partial x^2} + q, \qquad (0 \le x \le \ell, t \ge 0)$$

with inhomogeneous boundary conditions

$$w(0,t) = T_0, \quad w(\ell,t) = T_1,$$

and initial condition

$$w(x,0) = 2\sin(\pi x/\ell) + \sin(2\pi x/\ell) + \frac{q}{2k}x(\ell - x) + \frac{T_1 - T_0}{\ell}x + T_0,$$

where q, T_0 and T_1 are constants. Find the solution $w(x,t)$ (Hint: You can use the solution you have obtained in part (iii) to avoid duplicate calculation). ***(8 marks)***

4 (i) Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad u = u(x,t), \quad -\infty < x < \infty, \quad t \geq 0.$$

Given that the general solution to the wave equation is

$$u(x,t) = F(x+ct) + G(x-ct),$$

where $F(x)$ and $G(x)$ are two arbitrary functions, find the expressions for $F(x)$ and $G(x)$ such that $u(x,t)$ satisfies the initial conditions

$$u(x,0) = \frac{1}{1+x^2}, \qquad \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = 0 \ .$$

Hence write down the solution $u(x,t)$. ***(9 marks)***

(ii) The function $u(x,y)$ satisfies the Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \ ,$$

in the rectangular region $0 \leq x \leq a\,, 0 \leq y \leq b\,$, subject to boundary conditions:

$$u(0,y) = 0, \quad u(a,y) = 0, \quad u(x,0) = 0, \quad u(x,b) = -\sin(2\pi x/a). \tag{3}$$

Given that the solution is of the form $u(x,y) = -g(y)\sin(2\pi x/a)$ for some unknown function $g(y)$, find $g(y)$ and hence the solution $u(x,y)$. ***(12 marks)***

(iii) The function $w(x,y)$ satisfies the Laplace's equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0 \ ,$$

in the rectangular region $0 \leq x \leq a\,, 0 \leq y \leq b\,$, subject to boundary conditions:

$$w(0,y) = 0, \quad w(a,y) = 2\sin(\pi y/b), \quad w(x,0) = 0, \quad w(x,b) = -\sin(2\pi x/a).$$

The function $w(x,y)$ can be found as the sum of two functions $u(x,y)$ and $v(x,y)$ where $u(x,y)$ and $v(x,y)$ both satisfy the Laplace's equation subject to suitable boundary conditions. Find the boundary conditions for u and v so that the equations for u and v can be solved using the usual method of separation of variables. Do **NOT** attempt to solve the equations for u and v.

(4 marks)

End of Question Paper