The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2010–2011**

Vectors and Mechanics

2 **Hours**

Attempt all the questions. The allocation of marks is shown in brackets.

1 Two points P and Q have position vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ respectively, where

$$\boldsymbol{p} = 2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}, \qquad \boldsymbol{q} = 6\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}.$$

Find

(i) The vector $\overrightarrow{PQ}$;

(ii) The distance PQ;

(iii) The unit vector in the direction of $\overrightarrow{PQ}$;

(iv) The position vector of the mid-point of PQ;

(v) The vector equation of the line through the points P and Q. ***(10 marks)***

2 A particle of mass 2 kg is moving so that, at time t s, its position vector relative to a fixed origin O is given by

$$\boldsymbol{r}(t) = \left[t^2\boldsymbol{i} + 2t\boldsymbol{j} - 4t^2\boldsymbol{k}\right] \text{ m}.$$

Find:

(i) The velocity of the particle at time t s;

(ii) The acceleration of the particle at time t s;

(iii) The momentum of the particle at time t s;

(iv) The kinetic energy of the particle at time t s;

(v) The total force acting on the particle at time t s;

(vi) The work done by the force between $t = 0$ and $t = 2$. ***(12 marks)***

3 Given the vectors

$$\boldsymbol{u} = 2\boldsymbol{i} - 3\boldsymbol{j}, \qquad \boldsymbol{v} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}, \qquad \boldsymbol{w} = -4\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k},$$

by evaluating both sides, verify that

$$(\boldsymbol{u} \times \boldsymbol{v}) \times \boldsymbol{w} = (\boldsymbol{u}.\boldsymbol{w})\,\boldsymbol{v} - (\boldsymbol{v}.\boldsymbol{w})\,\boldsymbol{u}.$$

(9 marks)

4 Find the angle between the line

$$\boldsymbol{r} = \lambda\,(\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k})$$

and the normal to the plane

$$\boldsymbol{r}.\,(2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}) = 0.$$

(6 marks)

5 Particles A and B have velocity vectors $(3\boldsymbol{i} - 11\boldsymbol{j})$ m s^{-1} and $(5\boldsymbol{i} + \boldsymbol{j})$ m s^{-1} respectively. The velocity of a particle C relative to A is $(-2\boldsymbol{i} + 7\boldsymbol{j})$ m s^{-1}.
Find the velocity of C and the velocity of B relative to C. ***(4 marks)***

6 A projectile is launched from the origin O with initial speed V, at an angle θ above the horizontal. There is no air resistance.

If x and z are the horizontal and vertical displacements of the projectile relative to the origin O at time t after projection, write down expressions for x and z in terms of t, V, θ and the acceleration due to gravity g.

Hence show that

$$z = x\tan\theta - \frac{gx^2}{2V^2}\sec^2\theta.$$

(4 marks)

7 A particle of mass M is being pulled up a rough slope making an angle α with the horizontal, by a string which is parallel to the slope.

(i) Draw a clear force diagram showing all the forces on the block.

(ii) Find the components of the weight of the particle in the following directions:

(a) Parallel to the slope and directed up the slope;

(b) In the upwards perpendicular to the slope.

(iii) The coefficient of friction between the particle and the slope is μ.

Find the tension in the string if the block is moving up the slope with constant speed. ***(11 marks)***

8 Draw clear force diagrams for each of the following situations:

(i) A particle of mass m falling vertically downwards, with air resistance acting;

(ii) A car of mass m travelling at maximum speed without slipping round a rough bend of radius R which is at an angle α to the horizontal.

(5 marks)

9 A particle P performs simple harmonic motion about a centre O. At time t s the displacement of P from O is x m, where x satisfies the differential equation

$$\ddot{x} + 16x = 0.$$

Write down the general solution of this differential equation.

At time $t = 0$ the particle is at O and is given a velocity of 4 m s^{-1} in the positive x-direction.

(i) Find x in terms of t.

(ii) State the amplitude of the motion.

(iii) Find the magnitude and direction of the acceleration of P when it is at its greatest distance from O. ***(9 marks)***

10 Find, in the form $\boldsymbol{r}.\boldsymbol{n} = d$, the vector equation of the plane containing the lines

$$\boldsymbol{r} \times \boldsymbol{i} = -\boldsymbol{k} \qquad \text{and} \qquad \boldsymbol{r} \times \boldsymbol{j} = \boldsymbol{k}.$$

(10 marks)

11 A particle of mass m is attached to one end of a light inextensible string of length L. The other end of the string is fixed at a point O.
Initially the particle hangs at rest vertically below O. Air resistance can be ignored.

The particle is then set in motion horizontally with speed U, and starts moving in a vertical circle centred at O. At time t, the string makes an angle θ with the downward vertical from O.

(i) Draw a clear diagram showing all the forces acting on the particle.

(ii) Use the principle of conservation of energy to show that

$$\dot{\theta}^2 = \frac{U^2}{L^2} - \frac{2g}{L}\left(1 - \cos\theta\right),$$

where g is the acceleration due to gravity.

(iii) Find the tension in the string in terms of m, U, L, g and θ.

(iv) Show that the particle will complete a circular revolution about O if

$$U^2 > 5gL.$$

(20 marks)

End of Question Paper