The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS Spring semester 2009-2010

Applied Differential Equations 2 hours

Attempt all FOUR questions.

1 (i) Derive the first three non-zero terms of the Taylor series solution to the differential equation

$$\frac{dy}{dx} = x + y^2 - 2, \quad y(0) = 1 \ .$$

(5 marks)

(ii) **Given** that values of y obtained from this solution are $y(0.1) = 0.895$ and $y(0.2) = 0.78$, advance the solution to $x = 0.3$ by applying the Adams-Bashforth-Moulton method, given by

$$y_{n+1} = y_n + \frac{h}{2}\left[3f_n - f_{n-1}\right], \qquad (1)$$

$$y_{n+1} = y_n + \frac{h}{2}\left[f_{n+1} + f_n\right], \qquad (2)$$

with formula (1) as the predictor and formula (2) as the corrector. Here h is the step-size, y_i is an estimate of $y(x_i)$ and $f_i = f(x_i, y_i) = x_i + y_i^2 - 2$. Use $h = 0.1$, and work throughout correct to 4 decimal places. ***(7 marks)***

(iii) The following results were obtained by using the above Adams-Bashforth-Moulton method to solve the equation in part (i) as far as $x = 2$, with two different step-lengths:

h	$y(2)$
0.05	-0.496185
0.2	-0.495792

Use these data to estimate the step-length required to ensure that the global discretization error in $y(2)$ is smaller than 5×10^{-5}. You may assume that the Adams-Bashfort-Moulton method is of order 2. ***(4 marks)***

(iv) The predictor formula (1) can be written in the form

$$\sum_{j=0}^{2} \alpha_j y_{n+j} = h \sum_{j=0}^{2} \beta_j f_{n+j} \ .$$

Determine the values of α_j, β_j, $j = 0, 1, 2$. Show that the method is consistent. Write down the first characteristic polynomial for the method and show that the method is stable. ***(5 marks)***

(v) Apply the method given by formula (2) to the test equation $y' = \lambda y$, where λ is a constant, to show that

$$y_{n+1} = \frac{1 + \overline{h}/2}{1 - \overline{h}/2}\, y_n,$$

where $\overline{h} = h\lambda$. Hence show that the interval of absolute stability for the method is $(-\infty, 0)$.

(4 marks)

2 (i) Use Taylor series to derive the relation

$$y''(x_n) = \frac{y(x_{n+1}) - 2y(x_n) + y(x_{n-1})}{h^2} + O(h^2),$$

where $x_{n\pm1} = x_n \pm h$. ***(5 marks)***

(ii) Use the above relation and the relation

$$y'(x_n) = \frac{y(x_{n+1}) - y(x_{n-1})}{2h} + O(h^2)$$

to show that the differential equation

$$y''(x) + P(x)y'(x) + Q(x)y(x) = R(x)$$

may be approximated at $x = x_n$ by the following equation

$$(1 - \frac{1}{2}hP_n)y_{n-1} - (2 - h^2Q_n)y_n + (1 + \frac{1}{2}hP_n)y_{n+1} = h^2R_n\ ,$$

where $P_n = P(x_n)$, $Q_n = Q(x_n)$ and $R_n = R(x_n)$.

Use the above relation with $h = 0.25$ to show that the differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = 3,$$

with boundary conditions $y(1) = 2$, $y(2) = 4$ may be approximated by a system of linear algebraic equations $\boldsymbol{Ay} = \boldsymbol{b}$, where $\boldsymbol{b} = [-1.6125, 0.1875, -4.0981]^T$. Determine the elements of matrix $\boldsymbol{A}$ correct to 4 decimal places. **Do not attempt to solve these equations**. ***(14 marks)***

(iii) The following table contains grid-point values of two solutions $Y_1(x)$ and $Y_2(x)$ of a linear differential equation $d^2y/dx^2 = f(x, y, y')$ obtained using the fourth-order Runge-Kutta method. $Y_1(x)$ was determined using the initial conditions $y(1) = 1$, $y'(1) = 0$, and $Y_2(x)$ was obtained using $y(1) = 1$, $y'(1) = 1$.

x	1.25	1.5	1.75	2.0
$Y_1(x)$	1.07444	1.36280	2.02865	3.41281
$Y_2(x)$	1.36222	2.05472	3.35550	5.85592

Form a linear combination of these two solutions which is the numerical solution to the equation $d^2y/dx^2 = f(x, y, y')$ with boundary conditions $y(1) = 1$, $y(2) = 2$. Calculate the values of this solution at all the x-values in the table. ***(6 marks)***

3 (i) By using the change of independent variables $\nu = x + ct$, $\eta = x - ct$, show that the partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}\ , \quad u = u(x,t) \tag{3}$$

may be written as

$$\frac{\partial^2 u}{\partial \nu \partial \eta} = 0\ .$$

Hence determine the general solution of equation (3). ***(11 marks)***

(ii) Given that the general solution of equation (3) is in the form

$$u(x,t) = F(x+ct) + G(x-ct),$$

where F and G are arbitrary functions, use the initial conditions

$$u(x,0) = f(x), \quad -\infty < x < \infty\ ,$$
$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = g(x), \quad -\infty < x < \infty\ ,$$

to show that $F(x)$ and $G(x)$ satisfy

$$f(x) = F(x) + G(x)\ ,$$
$$g(x) = cF'(x) - cG'(x)\ ,$$

and hence show that

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)d\xi\ .$$

(14 marks)

4 (i) The function $u(x, y)$ satisfies Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \,, \quad u = u(x, y)$$

in the rectangular region $0 \le x \le a\,, 0 \le y \le b\,.$ Show that solutions of Laplace's equation of the form $X(x)Y(y)$ satisfy the relation

$$\frac{X''}{X} = -\frac{Y''}{Y} = \alpha \,,$$

where α is a constant. Explain why α is a constant. ***(4 marks)***

(ii) Assuming that $\alpha = s^2$ ($s \neq 0$ and s is a real number), show that

$$X(x) = A \cosh sx + B \sinh sx \quad \text{and} \quad Y(y) = C \cos sy + D \sin sy,$$

where A, B, C, and D are arbitrary constants. ***(4 marks)***

(iii) Find the corresponding solutions when $\alpha = 0$. ***(4 marks)***

(iv) Use the results in part (ii) and (iii) to determine the solution $u(x, y)$ of the Laplace's equation which satisfies the boundary conditions

$$u(0, y) = 0, \quad u(a, y) = u_0 \cos(3\pi y/b), \quad (0 \le y \le b)\,,$$
$$\left.\frac{\partial u(x, y)}{\partial y}\right|_{y=0} = 0, \quad \left.\frac{\partial u(x, y)}{\partial y}\right|_{y=b} = 0, \quad (0 \le x \le a)\,,$$

where u_0 is a constant. ***(13 marks)***

You may assume that only trivial solutions result from the choice $\alpha < 0$.

End of Question Paper