The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2009–2010**

MAS271 Methods for differential Equations **2 hours**

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

1 (i) Solve the equation

$$(1-t^2)\frac{d^2x}{dt^2} \;-\; 6t\frac{dx}{dt} - 4x = 0$$

near the ordinary point $t = 0$.

(14 marks)

(ii) Consider the system

$$\dot{x} = -y^3$$
$$\dot{y} = x^3$$

Using the Liapunov's theorem, show whether the equilibrium point is asymptotically stable or not.

(5 marks)

(iii) Consider the system

$$\dot{x} = -y + xy^2 + 2x^3$$
$$\dot{y} = x + x^2y + 3y^3$$

Try the Liapunov function $V(x, y) = ax^2 \;+\; by^2$ with a, $b > 0$ constants to be determined. Show using Liapunov's theorem if (0,0) is stable or not.

(6 marks)

2 The equations governing the two competing species x and y are

$$\dot{x} = x - x^2 - xy,$$
$$\dot{y} = \frac{3}{4}y - y^2 - \frac{x}{2}y,$$

where $x(\geq 0)$ and $y(\geq 0)$ are measured in appropriate units. Find the equilibrium points. ***(7 marks)***

What can you conclude about the nature of the equilibrium points. ***(10 marks)***

Sketch the trajectories in the xy-plane, indicating the direction of time, t, around the stable equilibrium point. ***(8 marks)***

3 (i) Write down the self-adjoint form of the differential equation

$$y'' - 4y' + \mu y = 0, \qquad 0 \leq x \leq 1.$$

The above differential equation is an eigenvalue problem with $y(0) = 0$ and $y'(1) = 0$.

State the orthogonality relation (in the form of a definite integral) satisfied by y_m and y_n the eigenfunctions associated with the eigenvalues μ_m and μ_n respectively. ***(5 marks)***

(ii) Show that if $\lambda = 25$ or $\lambda > 25$ the ordinary differential equation

$$y'' - 10y' + \lambda y = 0 \qquad (\lambda \text{ real constant}),$$

with

$$4y(0) = y'(0), \quad y(1) = 0,$$

has non-trivial solutions.
Show further that when $\lambda > 25$ the eigenvalues of λ giving non-trivial solutions are

$$\lambda_n = 25 + \mu_n^2 \qquad (n = 1, 2, 3, ...),$$

where μ_n is the nth positive root of $\tan\mu = \mu$.

Deduce that the eigenfunctions may be written as

$$y_0(x) = e^{5x}(1 - x)$$

$$y_n(x) = e^{5x}\left(\cos(\mu_n x) - \frac{\sin(\mu_n x)}{\mu_n}\right) \qquad (n \geq 1).$$

(12 marks)

 Question 3 continued on next page

3 (continued)

Verify the orthogonality relation

$$\int_0^1 e^{-10x} y_0(x) y_n(x) dx = 0 \qquad (n \geq 1).$$

(8 marks)

4 (i) Show that $y = 3 - 12x^2 + 4x^4$ is a solution of

$$y'' - 2xy' + 8y = 0$$

(2 marks)

Find a second solution in the form of a power series. ***(7 marks)***

Show that this series converges for all x and give its general term. ***(2 marks)***

Use your results to obtain the solutions of this differential equation satisfying:

(a) $y(0) = 1$, $y'(0) = 0$ and (b) $y(0) = 0$, $y'(0) = 1$. ***(4 marks)***

(ii) Show that there is only one solution of $xy'' + y = 0$ of the form

$$\sum_{n=0}^{\infty} a_n x^{n+\alpha} (a_o \neq 0)$$

and find its general term. ***(10 marks)***

5 (i) Find the value of n for which x^n is a solution of

$$x^2 y'' - xy' + y = 0.$$

Hence obtain the general solution. ***(10 marks)***

 Question 5 continued on next page

5 (continued)

(ii) For the following equations find the equilibrium points, investigate the nature of stability at these points and sketch the phase portrait:

$\dot{x} = 2x(1 - x - y)$, $\dot{y} = y(1 - 2y - 2x)$ (competing species) ***(15 marks)***

End of Question Paper