The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2009–10

Continuity and Integration — **2 hours**

Answer ***four*** *questions. If you answer more than four questions, only your best four will be counted.*

1 (i) Determine whether the sequences, whose nth terms are given below, converge or diverge. Give brief, precise reasons in all cases, and state the limits if they exist.

$$x_n := \frac{\sqrt{3n}+7}{\sqrt{n}+5}\,, \qquad y_n := \begin{cases} \left(\frac{1}{6}\right)^{\frac{1}{n}} & n \text{ even} \\ \left(\frac{1}{6}\right)^{n} & n \text{ odd} \end{cases}\,,$$

$$z_n := \frac{n^2 2^n + 3^n}{4^n + 5^n}\,, \qquad w_n := \cos\left(\frac{1}{n}\right)\,.$$

(12 marks)

(ii) Give the *formal* definition of the notion of a sequence of real numbers *converging to a limit*. Use this definition to prove that the sequence $\left(\frac{1}{\sqrt{n}}\right)$ converges to a limit. Use the definition of a limit to show that if c is a real constant and (x_n) is a sequence with $x_n \to x$ then $cx_n \to cx$.

(13 marks)

2 State which of the statements below are true and which are false. Prove those that are true, and provide counter examples with an explanation for those that are false. Theorems proved in lectures may be used without proof, provided that they are precisely stated.

(a) Every set with a minimum element is bounded below.

(b) Every set which is bounded below has a minimum element.

(c) Every sequence which is bounded above has a convergent subsequence.

(d) An increasing sequence cannot contain a strictly decreasing subsequence.

(e) A convergent sequence with a strictly increasing subsequence cannot contain a strictly decreasing subsequence.

(f) If E is a set of rational numbers, with a maximum but no minimum, then the difference $\sup E - \inf E$ must be rational. ***(25 marks)***

3 (i) Define what it means for a real-valued function to be *continuous*. For each of the following, give an example of such a function and sketch its graph (you do not need to prove that it has the required property):

(a) an *unbounded* continuous function $f\colon [-1,0) \cup (0,1] \to \mathbb{R}$ which does *not* extend to a continuous function on $[-1,1]$;

(b) a *bounded* continuous function $g\colon [-1,0) \cup (0,1] \to \mathbb{R}$ which does *not* extend to a continuous function on $[-1,1]$;

(c) a differentiable function $h\colon [-1,0) \cup (0,1] \to \mathbb{R}$ which extends to a continuous function on $[-1,1]$ but does not extend to a differentiable one. ***(13 marks)***

(ii) State Rolle's Theorem.

Suppose that $k\colon [a,b] \to \mathbb{R}$ and $l\colon [a,b] \to \mathbb{R}$ are continuous functions which are differentiable on (a,b), such that $k(a) \neq k(b)$ and such that there is *no* $t \in (a,b)$ with $k'(t) = l'(t) = 0$. Show that there is a $c \in (a,b)$ such that

$$\frac{l(b) - l(a)}{k(b) - k(a)} = \frac{l'(c)}{k'(c)}.$$

Hint: consider the function $h(t) := \big(l(b) - l(a)\big)k(t) - \big(k(b) - k(a)\big)l(t)$.

For k and l as above, consider the curve given by the parametric equations $x = k(t)$, $y = l(t)$ for $t \in [a,b]$. If $A = \big(k(a), l(a)\big)$ and $B = \big(k(b), l(b)\big)$ are the two end points of the curve then show that there is a point on the curve whose tangent line is parallel to the line AB.

(12 marks)

4 (i) If f is a real-valued function, give the definition of the *derivative* of f. Use the definition to find the derivative of $f(x) := x^n$ where n is a non-negative integer. ***(7 marks)***

(ii) Show from first principles that if f is a non-zero, differentiable function and g is defined by $g(x) := \dfrac{1}{f(x)}$ then g is also differentiable; give a formula for the derivative $g'(x)$. ***(5 marks)***

(iii) Use the first two parts of the question to deduce the derivative of $g(x) = x^n$ where n is a *negative* integer. ***(4 marks)***

(iv) For a positive integer m, let h the real-valued function with the maximum possible domain given by $h(x) := x^{1/m}$. What is the domain of h? State the formula for the derivative of the inverse of a function. Use this to find the derivative of h. ***(7 marks)***

(v) Indicate in one or two sentences how you would go on to obtain the formula

$$\frac{d}{dx}(x^p) = px^{p-1}$$

where p is any *rational* number. ***(2 marks)***

5 Fix a real number $b > 0$. Consider the function $f\colon [0, b] \to \mathbb{R}$ defined by $f(x) := b - x$, and the partition $P_N := \{0 = x_0 < \cdots < x_N = b\}$ of the interval $[0, b]$ into N equally sized pieces, so $x_i = \frac{ib}{N}$.

(i) Define what is meant by the *lower sum* $L_{P_N}(f)$. Draw a picture to illustrate the lower sum, indicating the relevant features. Find an explicit formula for the lower sum, simplifying your answer as much as possible.

Hint: You may use the formula $\sum_{i=1}^{N} i = \frac{1}{2}N(N+1)$. ***(12 marks)***

(ii) Define what it means for a real-valued function to be Riemann integrable. ***(3 marks)***

(iii) Prove that the function f is Riemann integrable on $[0, b]$, and give the value of the Riemann integral. ***(7 marks)***

(iv) Explain how the answer is in agreement with elementary geometry calculation for the area under the graph of the function. ***(3 marks)***

End of Question Paper