The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2009–10

MECHANICS — 2 hours

Answer four *questions. You are advised* not *to answer more than four questions: if you do, only your best four will be counted.*

1 (i) Calculate $\int\int_D \left(x^2 + xy^3\right)\, dxdy$ where D is the rectangular region $0 \leq x \leq 1, \quad 1 \leq y \leq 2.$ ***(10 marks)***

(ii) Sketch the region R bounded by $x = 0$, $y = 0$ and $2y + x = 2$, and evaluate

$$I = \int\int_R x(1-y)\, dxdy.$$

by integrating first with respect to x . ***(9 marks)***

Confirm your answer by also integrating first with respect to y. ***(6 marks)***

2 A uniform lamina of mass m is bounded by the curve with equation

$$y = \frac{x^2}{a^2}(a-x),$$

and that part of the x-axis between $x = 0$ and $x = a$, where a is a positive constant. You are **given** that the area of the lamina is $\frac{1}{12}a^2$.

(i) By considering thin strips of thickness δx parallel to Oy find the position of the centre of mass G of the lamina. ***(12 marks)***

(ii) Show that the moment of inertia of the lamina about the axis Oy is $\frac{2}{5}ma^2$. ***(7 marks)***

(iii) The lamina is rotating with angular speed ω about an axis through G parallel to Oy. What is its kinetic energy? ***(6 marks)***

3 (i) Plane polar coordinates (r, θ), with corresponding unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$, are defined by $x = r\cos\theta$, $y = r\sin\theta$, where (x, y) are Cartesian coordinates. Show that

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}},$$

$$\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}.$$

(10 marks)

(ii) A particle of mass m moves in a plane in a force field $\mathbf{F}$ given by

$$\mathbf{F} = \frac{2P\cos\theta}{r^3}\hat{\mathbf{r}} + \frac{P\sin\theta}{r^3}\hat{\boldsymbol{\theta}},$$

where (r, θ) are plane polar coordinates and P is a constant.

(a) Using the results of part (i) and the energy conservation principle, show that the total energy E (i.e. $E = T + V$ where T and V are the kinetic and potential energies respectively) during the particle's motion is given by

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{P\cos\theta}{r^2}.$$

(You may assume that $\nabla V = \dfrac{\partial V}{\partial r}\hat{\mathbf{r}} + \dfrac{1}{r}\dfrac{\partial V}{\partial \theta}\hat{\boldsymbol{\theta}}$.) ***(8 marks)***

(b) A quantity β is defined by

$$\beta = \frac{1}{2}mr^4\dot{\theta}^2 + P\cos\theta.$$

Show, using Newton's second law of motion and the results of part (i), that β is constant during the particle's motion. ***(7 marks)***

4 (i) A particle of mass m moves in a plane with origin O, and its plane polar coordinates are $(r(t), \theta(t))$. It is subject to a force directed towards O of magnitude $mF(r)$. You are **given** that the radial and transverse components of Newton's Second Law reduce to:

$$\ddot{r} - r\dot{\theta}^2 = -F(r) \qquad (1)$$

$$r^2\dot{\theta} = h \qquad (2)$$

where h is a positive constant (and, in the usual notation, a dot over a variable denotes its time derivative). Make the substitution $u = r^{-1}$, and use (2) to show that

$$\dot{r} = -h\frac{du}{d\theta}$$

and deduce that (1) becomes

$$h^2\left(\frac{d^2u}{d\theta^2} + u\right) = u^{-2}F(u^{-1}). \qquad (3)$$

(11 marks)

(ii) Solve equation (3) for the case $F(r) = \mu r^{-2}$, where μ is a positive constant.

(8 marks)

Given that there is a point on the path at which $\dot{r} = 0$, show that, in the case when $F(r) = \mu r^{-2}$, the equation of the path can be taken to be

$$\frac{1}{r} = \frac{\mu}{h^2}(1 + e\cos\theta),$$

where e is a constant. ***(6 marks)***

5 A thin uniform rod AB of mass m and length $2a$ is free to rotate in a vertical plane about a smooth horizontal axis through A. **Given** that the moment of inertia about the axis is $\frac{4}{3}ma^2$, find the period T_0 of small oscillations about the equilibrium position with B vertically below A. ***(9 marks)***

A particle P of mass km is now attached to the rod at a point distance ax from A, and the new period of small oscillations is T_1. Show that

$$\frac{T_1}{T_0} = \left(\frac{1 + \frac{3}{4}kx^2}{1 + kx}\right)^{\frac{1}{2}}.$$

(5 marks)

Use this result to verify that there is a positive value of x, independent of k, for which $T_1 = T_0$ for all k. When P is at the mid-point of AB, what is the minimum possible value of $\frac{T_1}{T_0}$? ***(11 marks)***

End of Question Paper