The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2009–10

**Linear Mathematics for Applications** — **2 hours**

*Answer **four** questions. If you answer more than four questions, only your best four will be counted.*

**1** Let

$$A := \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 7 \end{pmatrix}$$

(i) Transform the matrix $A$ by a sequence of elementary row operations to a matrix in reduced row echelon form. ***(5 marks)***

(ii) What is the rank of $A$ ? ***(1 mark)***

(iii) What is the determinant of $A$ ? Justify your answer. ***(2 marks)***

(iv) Determine the general solution of the system of linear equations $AX = 0$, where $X := \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}$, and write your general solution in (column) vector form. ***(5 marks)***

(v) Let

$$\mathcal{N}_A := \left\{ v \in \mathbb{R}^4 : Av = 0 \right\}.$$

Show that the set $\mathcal{N}_A$ is a subspace of $\mathbb{R}^4$. Find two vectors which span the subspace $\mathcal{N}_A$, and prove that they are linearly independent. ***(6 marks)***

(vi) Let $v_1 = (1, 2, 1, 1)^T$, $v_2 = (1, 2, 2, 3)^T$, $v_3 = (1, 2, 3, 5)^T$, and $v_4 = (1, 2, 4, 7)^T$. Let $W = \mathrm{Sp}\{v_1, v_2, v_3, v_4\}$.

(a) What is the dimension of the subspace $W$ ? Justify your answer.

(b) Find a basis of $W$, and justify your answer.

***(6 marks)***

**2** Let

$$A := \begin{pmatrix} 1/2 & 1/4 & 0 \\ 1/2 & 1/2 & 1/2 \\ 0 & 1/4 & 1/2 \end{pmatrix}.$$

(i) Find the eigenvalues of $A$, and for each eigenvalue a corresponding eigenvector. ***(11 marks)***

(ii) Express the vector $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})^T$ as a linear combination of your three eigenvectors found in part (i). ***(4 marks)***

(iii) A car rental agency has rental locations in Aberdeen, Birmingham, and Chesterfield.

A customer may rent a car from any of the three locations and return a car to any of these locations.

Each month, half the cars rented from Aberdeen are returned to Aberdeen, and half are returned to Birmingham. Of the cars rented from Birmingham, half are returned to Birmingham, and one quarter each are returned to Aberdeen and Chesterfield. Of the cars rented from Chesterfield, half are returned to Chesterfield, and the other half returned to Birmingham.

Initially, the business has half of its cars in Aberdeen, and one quarter in each of Birmingham and Chesterfield. What proportion of the cars can be found in the three location after 3 months of business ? ***(10 marks)***

**3** (i) State whether or not each of the following statements is true in general.

(a) A subset, $V$, of $\mathbb{R}^n$, is a subspace if it contains the zero vector.

(b) The columns of an $n \times n$ matrix $A$ form a basis for the space $\mathbb{R}^n$ if the matrix $A$ is invertible.

(c) If the columns of a matrix $A$ are linearly independent, and $A \sim B$, then the columns of the matrix $B$ are also linearly independent.

***(6 marks)***

(ii) For each of the following subsets $L_i$ $(i = 1, 2, 3, 4, 5)$ of $\mathbb{R}^2$, determine, with justification, whether $L_i$ is a subspace of $\mathbb{R}^2$.

(a) $L_1 := \{(x, y)^T \in \mathbb{R}^2 \mid x \geq y\}$;

(b) $L_2 := \{(x, y)^T \in \mathbb{R}^2 \mid x + y = 0\}$;

(c) $L_3 := \{(x, y)^T \in \mathbb{R}^2 \mid x - y = 0\}$;

(d) $L_4 := \{(x, y)^T \in \mathbb{R}^2 \mid x^2 - y^2 = 0\}$;

(e) $L_5 := \{(x, y)^T \in \mathbb{R}^4 \mid x^3 - y^3 = 0\}$.

***(10 marks)***

(iii) Let

$$A := \begin{pmatrix} 4 & 5 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -3 \end{pmatrix}$$

(a) Show that the set

$$V = \{Av \mid v \in \mathbb{R}^4\}$$

is a subspace of $\mathbb{R}^3$. What is its dimension ?

(b) Find a basis for the subspace $V$, and justify your answer.

(c) Find the dimension of the subspace

$$\mathcal{N}_A := \left\{ v \in \mathbb{R}^4 : Av = 0 \right\}.$$

***(9 marks)***

**4** Let

$$A := \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 3 \\ 3 & 2 & 2 \end{pmatrix}.$$

(i) Find the eigenvalues of the matrix $A$, and for each eigenvalue write down a corresponding eigenvector. ***(11 marks)***

(ii) Write down a matrix $P$ and diagonal matrix $D$ such that $P^{-1}AP = D$. ***(4 marks)***

(iii) Find the solution of the system of linear differential equations

$$\begin{aligned} y_1' &= 2y_1 + y_2 + y_3 \\ y_2' &= y_1 + 3y_2 + 3y_3 \\ y_3' &= 3y_1 + 2y_2 + 2y_3 \end{aligned}$$

for which $y_1(0) = 1$, $y_2(0) = -2$, and $y_3(0) = 1$. ***(10 marks)***

**5** Let $Q(x, y, z)$ be the real quadratic form given by

$$Q(x, y, z) = x^2 + 2y^2 + z^2 + 2xy - 2xz - 4yz.$$

(i) Express $Q(x, y, z)$ as a sum of squares and negatives of squares of linearly independent linear forms. You should explain why your linear forms are linearly independent. ***(7 marks)***

(ii) Determine the rank and signature of the quadratic form $Q(x, y, z)$. ***(2 marks)***

(iii) Determine the nature of the quadric surface in $\mathbb{R}^3$ whose equation is $Q(x, y, z) = 1$. ***(2 marks)***

(iv) Let

$$A := \begin{pmatrix} 1 & 1 & -2 \\ 1 & 2 & -1 \\ -2 & -1 & 1 \end{pmatrix}.$$

Find an invertible $3 \times 3$ matrix $S$ such that $S^T AS =: D$ is a diagonal matrix with diagonal entries taken from the set $\{1, -1, 0\}$. You should explain why your $S$ is invertible, and you should exhibit your $S$ and $D$ clearly. ***(8 marks)***

(v) Consider the quadratic form $R(x, y, z) = x^2 + 2y^2 + z^2 + 2xy - 2yz$. Determine the maximum and minimum values in the set

$$K := \left\{ Q(a, b, c) : a, b, c \in \mathbb{R} \text{ and } a^2 + b^2 + c^2 = 1 \right\}.$$ ***(6 marks)***

**End of Question Paper**