The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2008–9**

MECHANICS **2 hours**

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

1 (i) Calculate $\int\int_D xy^2 \, dxdy$ where

(a) D is the rectangular region $0 \leq x \leq 2, \quad 0 \leq y \leq 1$.

(b) D is the triangular region $0 \leq x \leq 2, \quad 0 \leq y \leq \frac{x}{2}$. ***(10 marks)***

(ii) Four small weights are joined together by light, rigid rods in a tetrahedral formation. The four weights, of masses 1 kg, 2 kg, 3 kg and 4 kg are initially held in position at coordinates $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, respectively, where distance units are in metres.

(a) Calculate the coordinates of the location of the centre of mass of the four weights.

(b) Calculate the moment of inertia of the tetrahedron about the Oz axis.

(c) The tetrahedron is now fixed to a pivot at $(0, 0, 1)$ (i.e. with the 4 kg weight at the pivot point) and allowed to hang freely from the pivot point under gravity (which acts in the $-\mathbf{k}$ direction). What are the coordinates of the centre of mass now, assuming the tetrahedron is at rest? (Give your answer to 2 decimal places.) ***(15 marks)***

2 A uniform lamina of mass m has the shape of the portion of the first quadrant bounded by $x^2 + y^2 = a^2$, $x = 0$ and $y = 0$.

(i) Use integration to find the position of its centre of mass G. ***(10 marks)***

(ii) Let I_y be the moment of inertia of the lamina about the axis Oy, and I_z be its moment of inertia about the axis Oz. Use the perpendicular axes theorem to deduce that

$$I_y = \frac{1}{2} I_z.$$

Hence show that

$$I_y = \frac{1}{4} m a^2.$$

(12 marks)

(iii) The lamina is rotating with angular speed ω about the axis Oz. Find its kinetic energy. ***(3 marks)***

3 (i) A comet of mass m moves in a plane with origin O, and its plane polar coordinates are $(r(t), \theta(t))$. The Sun, which has mass M and is considered to be fixed at the origin, exerts a gravitational pull on the comet: this is a force directed towards O of magnitude GMm/r^2, where $G = 6.67 \times 10^{-11}\,\mathrm{kg^{-1}m^3s^{-2}}$. You are **given** that the radial and transverse components of Newton's Second Law reduce to:

$$\ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2} \qquad (1)$$

$$r^2\dot{\theta} = h \qquad (2)$$

where h is a positive constant (and, in the usual notation, a dot over a variable denotes its time derivative). Make the substitution $u = r^{-1}$, and use (2) to show that

$$\dot{r} = -h\frac{du}{d\theta}$$

and deduce that (1) becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{h^2}. \qquad (3)$$

(11 marks)

(ii) Show that the general solution for (3) is

$$u = \frac{GM}{h^2}\left(1 + e\cos(\theta - \theta_0)\right)$$

where e and θ_0 are constants, and explain why θ_0 can be set to zero. Give a geometric interpretation for e. ***(7 marks)***

(iii) At its closest approach to the Sun, the comet is at a distance $r = 5\times10^{10}$ m from the origin and its speed is $9 \times 10^4\,\mathrm{ms^{-1}}$. The mass of the Sun is $M = 2 \times 10^{30}$ kg. Calculate the value of h and the eccentricity of the comet's orbit. (Give your answers to 1 decimal place.) ***(7 marks)***

4 (i) A scalar field ϕ is given by

$$\phi = \frac{y}{(x^2 + y^2)} \ .$$

(a) Calculate $\nabla\phi$.

(b) Show that

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \ .$$

(13 marks)

(ii) A force $\mathbf{F} = E_0\mathbf{i}$, where E_0 is a constant, acts on a particle of mass m, causing it to accelerate in the x direction.

(a) Is this a conservative force? If so, find a potential function V for it.

(b) The particle moves from the origin to the point (a, a). Calculate the work done by the force on the particle during this motion. If the speed of the particle when it reaches (a, a) is v, calculate the intial speed of the particle when it is at the origin. Deduce a lower bound on v, in terms of E_0, m and a. ***(12 marks)***

5 (i) A uniform rod AB of mass m and length $2a$ can rotate freely in a vertical plane about a fixed horizontal axis through the end A. When it is at rest with B vertically below A, the end B is suddenly given a velocity V. You are given that the moment of inertia of the rod about the axis through A is $\frac{4}{3}ma^2$. Use the principle of conservation of energy to show that, when the rod has turned though an angle θ,

$$\dot{\theta}^2 = \frac{V^2}{4a^2} - \frac{3g}{2a}(1 - \cos\theta)$$

and hence find $\ddot{\theta}$ by differentiation. ***(13 marks)***

(ii) When B is vertically above A, the reaction on the axis at A is zero. By considering the component of this reaction along the rod, show that

$$V^2 = 16ga,$$

and verify that the component perpendicular to the rod is zero.

(12 marks)

End of Question Paper