The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2008–2009

Differential and difference equations — **2 hours**

Answer ***four*** *questions. You are advised* ***not*** *to answer more than four questions: if you do, only your best four will be counted.*

1 (i) Solve the separable differential equation

$$y^3 \frac{dy}{dx} = (y^4 + 1)\cos x, \quad y = y(x)$$

subject to the initial condition $y(\pi/2) = 1$. ***(10 marks)***

(ii) The equation which gives the variation in mass of an isotope in a radioactive decay process is

$$\frac{dm}{dt} = -km, \quad m = m(t), \quad k > 0$$

Solve this differential equation supposing that $m(t = 0) = m_0$. ***(2 marks)***

The radiocarbon dating procedure is based on the radioactive decay of one of carbon's isotopes. Carbon extracted from an ancient skull contained only one-sixth as much as carbon extracted from present-day bone. Taking into account that the half-life of the used carbon isotope is 5730 years, find the age of the skull. Work throughout with a precision of two decimal places. ***(7 marks)***

(iii) P_n denotes the number of binary operations which take place in part of a computer program when n data points are supplied. If P_n satisfies the difference equation

$$P_{n+2} - P_{n+1} = 2(P_{n+1} - P_n),$$

and the conditions $P_0 = 10$ and $P_3 = 24$, obtain an expression for P_n in terms of n. ***(6 marks)***

2 (i) Solve the differential equation

$$\frac{dy}{dt} - 2y = 4 - t, \quad y = y(t),$$

using the integrating factor technique. Find the value of the integration constant when the solution satisfies the condition $y(0) = -2$. ***(10 marks)***

(ii) Each year 1000 salmons are added in a creek to an existing population (in the first year, the existing population is zero). Of these salmon, 30% survive until the following year when a new population of a 1000 is added. How many salmon will be in the creek at the end of each year? What will be the population of salmon in the very far future? Solve the problem as a first order difference equation. ***(5 marks)***

(iii) Find the general solution of the equation

$$y'' - 4y = 2e^{2x}$$

where $y = y(x)$. ***(10 marks)***

3 (i) Solve the differential equation

$$y'' + 4y = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x, \quad y = y(x)$$

subject to the conditions $y(0) = 6/5$ and $y'(0) = 2$. *Hint: find a particular solution for each inhomogeneous term.* ***(16 marks)***

(ii) Two species of birds compete with each other according to the equations

$$\frac{dx}{dt} = x - 3x^2 - 3xy, \qquad \frac{dy}{dt} = y + \frac{y^2}{2} - \frac{xy}{2},$$

where the populations are described by the continuous functions $x(t)$ and $y(t)$ in appropriate units. Define the equilibrium level and verify that $x = 1/3$, $y = 0$ is an equilibrium level of the above system of equations. Show that this equilibrium level is unstable. ***(9 marks)***

4 (i) Solve the difference equation

$$u_{n+2} - 5u_{n+1} + 6u_n = n^2 + 3$$

subject to the conditions $u_0 = 1$ and $u_1 = 6$. ***(13 marks)***

(ii) Find the general solution $y = y(x)$ of the differential equation

$$\frac{dy}{dx} = \frac{y}{x} + \csc\left(\frac{y}{x}\right)$$

using the substitution $y(x) = z(x)x$, with $1/e \le x \le e$ (e is the base of the natural logarithm). Hence find that solution satisfying the condition $y(1) = \pi/2$. Using the integration constant derived previously, find the values of x for which $y = 0$? ***(12 marks)***

5 (i) Solve the difference equation

$$u_{n+2} + u_n = 0$$

(5 marks)

(ii) The interaction between three species is described by the system

$$\frac{dx}{dt} = x(-a + y - z), \quad \frac{dy}{dx} = y(c - x - z), \quad \frac{dz}{dx} = z(b + x - y)$$

with $x = x(t)$, $y = y(t)$, $z = z(t)$ and a, b, c are constants. Find all *five* equilibrium levels of this model. ***(7 marks)***

(iii) Solve the following Euler-Cauchy differential equation

$$x^2 y'' + 2xy' + 3y = 0, \quad y = y(x)$$

subject to the initial conditions $y(1) = 1$ and $y'(1) = 0$. ***(13 marks)***

End of Question Paper