The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2008–2009

Rings and Groups — **2 hours**

*Answer **Question 1** and three other questions. You are advised **not** to answer more than three of the questions 2 to 5: if you do, only your best three will be counted.*

1 (i) Use Euclid's algorithm to find the inverse of 25 in the ring $\mathbb{Z}_{47}$. ***(6 marks)***

(ii) Is each of the following rings an integral domain? Justify your answer briefly.

(a) $\mathbb{C}$

(b) $\mathbb{Z}_6$

(c) $\mathbb{Z}_{11}$

(d) $\mathbb{Z}_{11}[x]$

(e) $\mathbb{Z}[\sqrt{-5}]$ ***(8 marks)***

(iii) Write down all possible cycle types in S_4, together with the number of elements in S_4 of each type. Hence write down the class equation for S_4. Justify your answers. ***(11 marks)***

2 (i) (a) What are the units in the ring $\mathbb{Z}_{10}$? Justify your answer, and for each unit give its multiplicative inverse. ***(7 marks)***

(b) Show that the group of units of $\mathbb{Z}_{10}$ is cyclic. ***(4 marks)***

(ii) Let d be a square-free integer with $d \neq 1$. Recall that the norm of an element $r = a + b\sqrt{d}$ of $\mathbb{Z}[\sqrt{d}]$, where $a, b \in \mathbb{Z}$, is given by

$$N(a + b\sqrt{d}) = |a^2 - b^2 d|.$$

(a) Show that $\mathbb{Z}[\sqrt{-7}]$ has no element of norm 2. Hence show that any element of norm 4 or 8 is irreducible. ***(8 marks)***

(b) Write down an element of norm 8 in $\mathbb{Z}[\sqrt{-7}]$. Hence express 8 as a product of irreducible factors in $\mathbb{Z}[\sqrt{-7}]$ in two different ways, and deduce that $\mathbb{Z}[\sqrt{-7}]$ is not a unique factorisation domain. Justify your answer. ***(6 marks)***

3 (i) Are the following elements zero-divisors in the given rings? Justify your answers.

(a) $3 \in \mathbb{Z}$

(b) $3 \in \mathbb{R}$

(c) $3 \in \mathbb{Z}_n$ where $n = 123123123123123123123123123$

(d) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in \mathbf{Mat}_2(\mathbb{Z})$ ***(6 marks)***

(ii) (a) Let R be an integral domain. Prove that the units in the polynomial ring $R[x]$ are precisely the units in R. ***(8 marks)***

(b) Find a non-constant polynomial in $\mathbb{Z}_9[x]$ which is a unit, justifying your answer. ***(3 marks)***

(iii) Calculate the norm of $4 + i \in \mathbb{Z}[i]$. Hence exhibit a prime number that is not irreducible in $\mathbb{Z}[i]$. Does 13 have this property? Justify your answers carefully.

(8 marks)

4 (i) Let G be a group of order 10 with trivial centre.

(a) Find the class equation for G, justifying your answer. ***(5 marks)***

(b) Find the number of elements of order 5 in G. ***(6 marks)***

(c) Let $h \in G$ be an element of order 5. Let $H = \langle h \rangle$, the subgroup generated by h. Use the class equation to show that H is a normal subgroup of G. ***(4 marks)***

(ii) (a) Let B be a normal subgroup of a group A. Describe the quotient group A/B. (You need to specify what the elements are, how multiplication on A/B is defined, what the identity is, and what the inverse of a given element is, but you do not need to prove any of your assertions.) ***(5 marks)***

(b) Now let A be the cyclic group of order 4 with elements $1, a, a^2, a^3$. Let B be the subgroup generated by the element a^2. Show that the quotient group A/B is a cyclic group of order 2. ***(5 marks)***

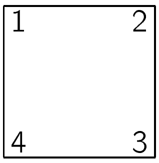
5 (i) Let G be a group of order 7.

(a) Prove that G must be cyclic. ***(4 marks)***

(b) Hence find the class equation for G, justifying your answer carefully. ***(6 marks)***

(ii) (a) State, without proof, the First Isomorphism Theorem for groups. ***(2 marks)***

(b) Consider a square with vertices labelled as shown.



Write D_4 for the group of symmetries of the square. Explain briefly how the action of D_4 on the vertices of the square gives rise to a homomorphism $f : D_4 \longrightarrow S_4$. ***(4 marks)***

(c) Find the kernel and image of the above homomorphism f, justifying your answer. ***(7 marks)***

(d) What does the First Isomorphism Theorem tell us in this case? ***(2 marks)***

End of Question Paper