The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2008-2009**

Vectors and Fluids **2 hours**

Answer ***four*** *questions. You are advised* ***not*** *to answer more than four questions: if you do, only your best four will be counted.*

1 (i) The vector field $\mathbf{u}$ has components $(-\Omega y, \Omega x, 0)$ where Ω is a constant. In the usual notation, (x, y, z) are the components of position vector $\mathbf{r}$. Find

$$(a)\nabla\cdot\mathbf{u};\quad (b)\nabla\times\mathbf{u};\quad (c)\nabla(\tfrac{1}{2}\mathbf{u}\cdot\mathbf{u});\quad (d)(\mathbf{u}\cdot\nabla)\mathbf{u}.$$

Verify that

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = -\Omega^2\vec{NP},$$

where $\vec{OP} = \mathbf{r}$ and N is the foot of the perpendicular from P to Oz. Recalling that $(\mathbf{u}\cdot\nabla)\mathbf{u}$ is the acceleration of a fluid particle when the velocity field is $\mathbf{u}$, explain, clearly but very briefly, why the last result can be written down without working. ***(13 marks)***

(ii) If, in standard notation, $r = \sqrt{(x^2 + y^2 + z^2)}$, show that

$$\frac{\partial r}{\partial x} = \frac{x}{r}.$$

If $V = xr^{-3}$, find ∇V and deduce that $\nabla^2 V = 0$.

(12 marks)

2 (i) Explain, briefly but clearly, why the i-component of $\nabla \times (\nabla \times \mathbf{F})$ is

$$\epsilon_{ijk}\frac{\partial}{\partial x_j}(\epsilon_{klm}\frac{\partial F_m}{\partial x_l}).$$

Given that

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl},$$

deduce, using suffix notation, that

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \boldsymbol{\cdot} \mathbf{F}) - \nabla^2 \mathbf{F}.$$

(10 marks)

(ii) Given that

$$L = \beta \begin{pmatrix} 1 & -\sqrt{3} & 2\sqrt{3} \\ 2\sqrt{3} & 2 & 0 \\ \sqrt{3} & -3 & -2 \end{pmatrix},$$

where β is a constant, find LL^T and $\det L$ in terms of β. If L is a transformation matrix for the rotation of axes from an undashed right-handed frame $Ox_1x_2x_3$ to a dashed right-handed frame $Ox_1'x_2'x_3'$, find the value of β.

Find the values of λ and μ such that L represents a rotation about the vector $(1, \lambda, \mu)$. ***(15 marks)***

3 Cylindrical polar coordinates (r, θ, z) are related to Cartesian coordinates (x, y, z) by

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

(i) Obtain expressions for δx and δy in terms of $r, \theta, \delta r, \delta\theta$, where $\delta x, \delta y, \delta r, \delta\theta$ are, in standard notation, small changes in x, y, r, θ respectively. Hence find $(\delta x)^2 + (\delta y)^2 + (\delta z)^2$ in terms of $r, \delta r, \delta\theta, \delta z$. Explain, giving clear but concise reasons, why your result shows that cylindrical polar coordinates are orthogonal, and write down expressions for h_1, h_2, h_3 in standard notation (where h_1, h_2, h_3 correspond to r, θ, z respectively). ***(10 marks)***

(ii) The vector field $\mathbf{F}$ is defined by

$$\mathbf{F} = (x^2 + y^2)z\mathbf{i} + z^3\mathbf{k}.$$

Verify that Gauss's Theorem, namely

$$\int_S \mathbf{F} \boldsymbol{\cdot} \mathbf{dS} = \int_V \nabla \boldsymbol{\cdot} \mathbf{F} dV,$$

holds when V is the finite cylinder bounded by $r = a$ (where r is a cylindrical polar coordinate) and the planes $z = 0, z = h$. ***(15 marks)***

4 The velocity potential ϕ in the two-dimensional irrotational flow of an incompressible fluid is defined by

$$\phi = \alpha \left(\frac{x}{x^2 + y^2} \right),$$

where (x, y) are Cartesian coordinates and α is a real constant. Find the velocity field $\mathbf{u}$ and verify that $\nabla \cdot \mathbf{u} = 0$. ***(11 marks)***

Given that the complex variable ζ is equal to $x + iy$, where, in standard notation, $i^2 = -1$, show that ϕ is the real part of $\left(\frac{\alpha}{\zeta}\right)$. Let ψ be the imaginary part of $\left(\frac{\alpha}{\zeta}\right)$. Find ψ, and show that it is the stream function for this flow. ***(14 marks)***

5 With respect to cylindrical polar coordinates (r, θ, z) (with $r \geq 0$, $0 \leq \theta \leq 2\pi$), the velocity potential ϕ in the steady irrotational flow of an incompressible fluid of uniform density ρ in the region $r \geq a$ is given by

$$\phi = U \left(r + \frac{a^2}{r} \right) \cos\theta - 2aU\theta,$$

where U and a are constants. You may assume that this ϕ satisfies Laplace's equation, but state why this is necessary. ***(2 marks)***

(i) Given that

$$\nabla\phi = \frac{\partial\phi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\boldsymbol{\theta}},$$

find the velocity field $\mathbf{u}$. ***(3 marks)***

(ii) Show that $\mathbf{u} \cdot \mathbf{n} = 0$ on $r = a$ (where $\mathbf{n}$ is the unit outward normal to the cylinder $r = a$) and state the behaviour of $\mathbf{u}$ for large r. ***(4 marks)***

(iii) Describe the flow in words in no more than two sentences. ***(4 marks)***

(iv) Show that $|\mathbf{u}| = 0$ only when $r = a$, $\theta = 3\pi/2$. ***(8 marks)***

(v) Given that there is no gravity acting and that the pressure p tends to p_∞ as $r \to \infty$ (where p_∞ is a constant), deduce that p has a maximum value of $p_\infty + \rho U^2/2$ and state where this is attained. ***(4 marks)***

End of Question Paper