The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2008–9

Advanced Calculus — 2 hours

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

1 (i) The random variables X and Y have joint density function

$$f_{X,Y}(x,y) = \begin{cases} e^{-x}e^{-y} & \text{if } x \geq 0 \text{ and } y \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the probability $P(X+Y \leq \alpha)$, where α is a fixed positive number. ***(9 marks)***

(ii) Let $\omega = (y^2 - \sin x \sin y)\, dx + (2xy + \cos x \cos y)\, dy$.

(a) Show, *without* finding a potential function, that ω is an exact differential.

(b) Now find a potential function f for ω.

(c) Evaluate the line integral $\int_\gamma \omega$, where γ is any path starting at $x = 0$, $y = 0$ and finishing at $x = \pi/2$, $y = 1$. ***(12 marks)***

(iii) A function $F(x)$ is defined for $x > 0$ by the formula

$$F(x) = \int_0^x e^{-xt^2}\, dt.$$

Write down an expression for the derivative $\dfrac{dF}{dx}$. (Do not attempt to evaluate any integrals.) ***(4 marks)***

2 (i) Green's Theorem states that

$$\int_C P dx + Q dy = \int\int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dx\, dy \ .$$

(a) Explain carefully what C and D are, and state any conditions needed for the theorem's validity.

(b) Evaluate **both** sides of the theorem for $P = xy$, $Q = x^2y^3$ and C the positively oriented triangular path connecting vertices $(0, 0)$, $(1, 1)$ and $(0, 1)$. Hence show explicitly that in this case the two sides of the theorem are equal. ***(16 marks)***

(ii) Let C be the ellipse parametrised by

$$x = a\cos t,\ y = b\sin t,\ 0 \le t \le 2\pi.$$

Use a line integral around the ellipse to calculate the area enclosed within it. ***(9 marks)***

3 Let f be the periodic function with period 2π such that

$$f(x) = \begin{cases} \cos x, & \text{for } 0 \le x < \pi; \\ -\cos x, & \text{for } -\pi \le x < 0. \end{cases}$$

(i) Sketch the graph of the function f. Explain why the Fourier series of f contains only *sine* terms. ***(5 marks)***

(ii) Show that the Fourier series for f is

$$\frac{8}{\pi} \sum_{m=1}^{\infty} \frac{m}{4m^2 - 1} \sin(2mx) \ .$$

(15 marks)

(iii) Deduce that

$$\frac{1}{4 \cdot 1^2 - 1} - \frac{3}{4 \cdot 3^2 - 1} + \frac{5}{4 \cdot 5^2 - 1} - \cdots = \frac{\pi\sqrt{2}}{16} \ .$$

(5 marks)

4 (i) If X is a random variable, define the probability generating function $G_X(s)$. In the case that X is a discrete random variable taking only the values $0, 1, 2, \ldots$, show that the mean of X (if it exists) is given by $E[X] = G_X'(1)$.

(5 marks)

(ii) Suppose that
$P(X = k) = (3/4)(1/4)^k$ for $k \geq 0$.

(a) Show that $G_X(s) = \dfrac{(3/4)}{1 - (s/4)}$, and find $E[X]$.

(b) Let Y be another random variable, independent of X, but with the same distribution. What is the generating function of $X + Y$? Calculate $P(X + Y = 2)$.

(10 marks)

(iii) Recall the Fourier transform $\hat{f}(s) = \int_{-\infty}^{\infty} f(t)e^{-ist}\,dt$ (when it exists), and the Fourier inversion formula

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(x)e^{itx}\,dx$$

(valid under certain conditions).

(a) Calculate the Fourier transform of $e^{-|t|}$.

(b) Hence show that

$$\int_0^{\infty} \frac{\cos st}{1+s^2} ds \;=\; \frac{\pi}{2}e^{-|t|}\ .$$

(10 marks)

5 (i) Find and classify the critical points of the function

$$f(x, y) = x^3 - 3x + xy^2.$$

(13 marks)

(ii) Use the method of Lagrange multipliers to find the stationary points of the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

subject to the constraint

$$3x - 2y + z - 4 = 0\ .$$

Give a geometrical interpretation. State whether any stationary points found are minima, maxima or saddle points of f subject to the constraint.

(12 marks)

End of Question Paper