The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2008-2009**

Topics in Number Theory **2 hours**

Answer ***four*** *questions. If you answer more than four questions, only your best four will be counted.*

No credit will be given for solutions which rely solely on the use of a calculator. Your solutions should give enough details to make it clear how you arrived at the answer.

1 (i) Determine the remainder when 2009^{2009} is divided by 95. ***(7 marks)***

[No credit will be given for a solution which does not use Fermat's Little Theorem.]

(ii) In the RSA cryptosystem, you publish $(n, e) = (187, 7)$ and receive 13. Decode it. ***(13 marks)***

(iii) Determine the remainder when 28! is divided by 31. ***(5 marks)***

[No credit will be given for a solution which does not use Wilson's Theorem.]

2 (i) State *The Law of Quadratic Reciprocity* and use it to determine whether the congruence

$$x^2 + 3x + 31 \equiv 0 \pmod{103}$$

has a solution. If it has, solve it. ***(16 marks)***

(ii) Let p be a prime number greater than 2. Show that the congruence

$$x^4 + 3x^2 + 2 \equiv 0 \pmod{p}$$

has a solution if and only if either $\left(\frac{-1}{p}\right) = 1$ or $\left(\frac{-2}{p}\right) = 1$. Deduce that it has no solution if and only if $p \equiv 7 \pmod{8}$. ***(9 marks)***

3 (i) Find the sum of the positive divisors of 24000. *(6 marks)*

(ii) For each of the numbers given, find a prime number which divides it:

(a) $2^r - 1$ where r is an even natural number,

(b) $2^s + 1$ where s is an odd natural number,

(c) $2^{121} - 1$

(d) $2^{124} + 1$. *(9 marks)*

(iii) Factorize 640^4 into its prime factors and hence show that $2^{32}+1$ is divisible by 641. *(5 marks)*

(iv) State a formula which gives all even perfect numbers, and show that every even perfect number is the sum of the first q natural numbers for some prime number q. (You may quote the formula

$$1 + 2 + \ldots + n = \tfrac{1}{2}n(n+1)$$

for the sum of the first n natural numbers.) *(5 marks)*

4 Let x, y, z be a primitive Pythagorean triple. Show that one of x, y is even and the other is odd, and state formulae which describe all primitive Pythagorean triples with x even. *(5 marks)*

(a) Find all primitive Pythagorean triples which include 333. *(10 marks)*

(b) Find *two* non-primitive Pythagorean triples which include 333. *(2 marks)*

(c) Is there a primitive Pythagorean triple which includes 338? *(2 marks)*

(d) Is there a non-primitive Pythagorean triple which includes 338? *(2 marks)*

(e) Is there a Pythagorean triple in which two of the numbers are squares? Justify your answer. *(4 marks)*

5 (i) Express $\sqrt{15}$ as a continued fraction, find a convergent which differs from $\sqrt{15}$ by less than 10^{-5} and find a solution of the Pell equation $x^2 - 15y^2 = 1$ in positive integers with $x > 100$. *(18 marks)*

(ii) Let $\alpha = \dfrac{1+\sqrt{5}}{2}, \beta = \dfrac{1-\sqrt{5}}{2}$. Prove that $(\alpha^n + \beta^n)(\alpha - \beta) = \alpha^{n+1} - \beta^{n+1} + \alpha^{n-1} - \beta^{n-1}$ and deduce that, for all $n > 1, f_{n+1} + f_{n-1} = \dfrac{f_{2n}}{f_n}$, where (f_n) denotes the Fibonacci sequence. *(7 marks)*

End of Question Paper