The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2007-2008

NUMERICAL LINEAR ALGEBRA — Two hours

Marks will be awarded for your best FOUR answers

1 (i) (a) Define the condition number, $\mathcal{K}(A)$, of a non-singular square matrix A in any norm, $\|\cdot\|$. ***(2 marks)***

(b) If a matrix A is poorly conditioned, then the possibility exists that $A+\delta A$, for some small perturbation δA, is *singular.* By considering the equation

$$(A+\delta A)\mathsf{x} = 0$$

for nonsingular A, show that $A+\delta A$ *cannot* be singular if $\left\|A^{-1}\right\| \|\delta A\| < 1$. ***(7 marks)***

(ii) We wish to solve $A\mathsf{x} = \mathbf{b}$ where $\mathbf{b}$ is known exactly but where A is subject to an uncertainty δA. Thus, in effect we necessarily solve

$$(A+\delta A)(\mathsf{x}+\delta\mathsf{x}) = \mathbf{b}.$$

Assuming that $\left\|A^{-1}\right\| \|\delta A\| < 1$, then prove that the relative error in x satisfies

$$\frac{\|\delta\mathsf{x}\|}{\|\mathsf{x}\|} \leq \frac{\mathcal{K}(A)}{\left\{1-\mathcal{K}(A)\frac{\|\delta A\|}{\|A\|}\right\}} \frac{\|\delta A\|}{\|A\|}.$$

(9 marks)

(iii) Given

$$A \approx \begin{bmatrix} 1.0000 & 0.2500 & 0.1111 & 0.0625 \\ 0.2500 & 0.1111 & 0.0625 & 0.0400 \\ 0.1111 & 0.0625 & 0.0400 & 0.0278 \\ 0.0625 & 0.0400 & 0.0278 & 0.0204 \end{bmatrix} \quad \text{and} \quad \|\delta A\| \approx 0.00002,$$

(a) then write ***scilab*** code to determine bounds on $\|\delta\mathsf{x}\|_\infty/\|\mathsf{x}\|_\infty$. ***(4 marks)***

(b) Given this calculation yields $\|\delta\mathsf{x}\|_\infty/\|\mathsf{x}\|_\infty \leq 1.4$, comment on the usefulness of solutions computed for this system. ***(3 marks)***

2 (i) Given $m+1$ data points $(x_j,\ f_j)$, $j = 0, 1, \ldots, m$, where the x_j values are all distinct, derive the normal equations for determining the least square fit to the data by the function

$$P_n(x) = \sum_{i=0}^{n} \alpha_i x^i$$

without weights.

(10 marks)

(ii) Hence, write down the normal equations for the best quadratic fit, and write ***scilab*** code to compute this fit for the data

x_j	0.1000	0.2000	0.3000	0.4000	0.5000
f_j	0.3528	1.6896	2.2264	1.8432	0.4200 .

(10 marks)

(iii) Given that the parabola of the previous part is given by

$$P_2(x) = -46.0\,x^2 + 27.9\,x - 2.0$$

then calculate the residuals and comment briefly on whether or not you think the degree of the polynomial should be increased. ***(5 marks)***

3 (i) (a) Verify, by direct expansion, that the matrix equation

$$A^T A\alpha = A^T \mathbf{f} \tag{1}$$

where

$$A = \begin{pmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & x_m & x_m^2 & \dots & x_m^n \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_m \end{pmatrix}, \quad \alpha = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix}$$

is equivalent to the normal equations arising from deriving a least squares fit to the data $(x_j,\ f_j),\ j = 0..m$. ***(3 marks)***

(b) Verify that, if P is an orthogonal matrix, then equations (1) remain as the normal equations when the transformed residuals

$$\hat{\mathbf{r}} \equiv P\mathbf{r} = P(A\alpha - \mathbf{f})$$

are minimized ***(3 marks)***

(ii) The Householder Reflection matrix is given by

$$P = \left(I - 2\frac{\mathbf{w}\mathbf{w}^T}{\mathbf{w}^T\mathbf{w}} \right).$$

Given the vector $\mathbf{x}^T = (x_0, x_1, ..., x_m)$ and the relation

$$\hat{\mathbf{x}} = P\mathbf{x},$$

where P, above, is based upon the vector

$$\mathbf{w} = \mathbf{x} + sign(x_0)\,||\mathbf{x}||_2\,\mathbf{e}_0,$$

where $\mathbf{e}_0$ is the first column of the $(m+1)\times(m+1)$ identity matrix, show that

$$\hat{\mathbf{x}} = -sign(x_0)||\mathbf{x}||_2\,\mathbf{e}_0.$$

(7 marks)

(iii) Apply a Householder Reflection on the column vector $\mathbf{x} = (1, 1, 1, 1, 1)^T$, according to the theory immediately above, showing that the result, $\hat{\mathbf{x}}$, is proportional to $(1, 0, 0, 0, 0)^T$. Work correct to four decimal places throughout. ***(8 marks)***

(iv) We wish to apply a single Householder reflection to the matrix

$$A = \begin{bmatrix} 1 & 2.2 \\ 1 & 2.4 \\ 1 & 2.6 \\ 1 & 2.8 \\ 1 & 3.0 \end{bmatrix}.$$

Write ***scilab*** code to accomplish this task.

(4 marks)

4 (i) The real symmetric matrix A has eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ satisfying

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq ... \geq |\lambda_n| > 0$$

with corresponding linearly independent eigenvectors $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n$ which can be supposed normalized so that the largest element of each one is unity**.**

(a) Write down the Power iteration for finding the dominant eigenvalue and its eigenvector. ***(3 marks)***

(b) Prove that, given the assumptions above, the iterates converge to the dominant eigenvalue and its eigenvector. ***(9 marks)***

(ii) Show that if a matrix A has an eigenvalue λ, then the matrix $B \equiv A - pI$ has an eigenvalue $\lambda - p$. ***(2 marks)***

(iii) The matrix

$$A = \begin{pmatrix} 4.0 & 2.1 & 0.1 \\ 2.1 & 5.0 & 2.1 \\ 0.1 & 2.1 & -4.1 \end{pmatrix}$$

has eigenvalues given approximately by -4.6, 2.5 and 6.9 .

(a) Which of the three values 5, -5 or -7 is a suitable choice for p for the determination of the eigenvalue nearest to -4.6 using the Power method with shift of origin? State the reason for your choice. ***(2 marks)***

(b) Use your chosen value of p together with $\mathbf{z}_0^T = (0, -0.2, 1.0)$ to compute **one** further estimate to the eigenvalue near -4.6 and its corresponding normalized eigenvector. Work correct to four significant figures, stating clearly your eigenvalue estimate. ***(2 marks)***

(c) Write ***scilab*** code that will perform ten iterations of the Power method with shift of origin for the determination of the eigenvalue near -4.6 and its corresponding normalized eigenvector. ***(7 marks)***

5 (i) The linear system

$$A\mathsf{x} = \mathbf{b},$$

where A is an $n \times n$ matrix of known coefficients, $\mathbf{b}$ is an $n \times 1$ column vector of known values and x is an $n \times 1$ column vector of unknowns, can be rearranged in arbitrarily many ways in the form

$$\mathsf{x} = H\mathsf{x} + \mathbf{d}$$

which can subsequently be used to define the iteration

$$\mathsf{x}^{(k+1)} = H\mathsf{x}^{(k)} + \mathbf{d} \qquad (2)$$

where H is some $n \times n$ matrix and $\mathbf{d}$ is a column vector of known values.

(a) Derive a sufficient condition, written in terms of a norm of the matrix H, that the iteration (2) above will give a sequence of iterates convergent to x. ***(5 marks)***

(b) Starting from $A\mathsf{x} = \mathbf{b}$, derive the Jacobi iteration, and hence prove that **strict diagonal dominance** of the matrix A is sufficient to guarantee the convergence of the method. ***(8 marks)***

(ii) (a) Use the Gauss-Seidel iterative method to obtain two successive approximations to the solution of the system $A\mathsf{x} = \mathbf{b}$ where

$$A = \begin{pmatrix} 10 & 4 & 1 \\ 4 & 10 & 3 \\ 1 & 3 & 10 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix},$$

using $\mathsf{x}^{(0)} = (1, 1, 1)^T$ as the starting vector and working correct to four significant figures. ***(6 marks)***

(b) Write ***scilab*** code with implements the Gauss-Seidel method to generate an approximate solution to the system $A\mathsf{x} = \mathbf{b}$ defined above that will iterate until $||A\mathsf{x} - \mathbf{b}||_\infty \leq 10^{-5}$. ***(6 marks)***

End of Question Paper