The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn 2015-2016**

Financial Mathematics 2 **hours and** 30 **minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

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1 Consider the following four bonds with face value of £100:

Time to maturity (in years)	Annual interest (paid every 6 months)	Bond price (in £)
0.5	0%	99.005
1.	6%	102.926
1.5	0%	95.313
2.	4%	100.93

(i) Find the 0.5-year spot interest rate. ***(2 marks)***

(ii) Use the bootstrap method to find the 1-year and 2-year spot interest rates. ***(8 marks)***

(iii) Suppose that you are offered by a risk free institution the opportunity to deposit or borrow £1,000,000 in one year for a period of one year earning an interest rate of 3.75%. Describe in detail an arbitrage opportunity available to you. ***(12 marks)***

(iv) Consider a forward contract to deliver the 2-year bond listed in the table above, with delivery date in 1 year. (Assume that the bond is delivered immediately after the payment of its second coupon.) What is the correct forward price for this contract? ***(3 marks)***

2 (i) (a) Consider the following two portfolios consisting of European options on the same underlying asset and same expiration date:

- Porfolio A contains 6 put options with strike price 20, 6 put options with strike price 40 and a short position on 12 put options with strike price 30.
- Porfolio B contains 3 call options with strike price 10, 2 call options with strike price 60 and a short position on 5 call options with strike price 30.

Sketch the graphs of the payoff functions of these portfolios at expiration. ***(6 marks)***

(b) Let p_{20}, p_{30} and p_{40} be the spot prices of the put options in part (a) with strike prices 20, 30 and 40, respectively. Let c_{10}, c_{30} and c_{60} be the spot prices of the call options in part (a) with strike prices 10, 30 and 60, respectively. Describe an inequality satisfied by these spot prices and explain in detail why it holds. ***(7 marks)***

(ii) The price of a stock which pays no dividends is currently £100 and over the next two 1-year periods the price will either increase by 10% or decrease by 10%. An option which expires in two years entitles the owner to the *maximum* of the share prices observed at times 0, 1 and 2 years. Interest rates for all periods are 5%.

(a) Use a tree to find the spot price of the option. (Hint: note that the option's payoff when the stock price goes up and then down is different from the option's payoff when the stock price goes down and then up). ***(8 marks)***

(b) Consider a option similar to the one above on the same underlying asset which additionally gives the owner the right to sell the option for £100 at times 0, 1 and 2 years. Find the spot price of this new option. ***(4 marks)***

3 (i) Explain the principle of risk-neutral valuation. ***(4 marks)***

(ii) State Ito's Lemma. ***(4 marks)***

(iii) Consider a derivative on a stock which entitles the holder to one payoff in T years; the amount of this payoff is £1 if the stock price S_T in T years is between a and b for some positive numbers a and b such that $a < b$, and zero otherwise. Let S be the price of the stock and assume, as usual, that S follows the process

$$dS = \mu S dt + \sigma S dB$$

for constants μ and $\sigma > 0$ and where B is a Brownian motion. Assume further that all interest rates are constant and equal to r.

(a) Use Ito's Lemma to find $d(\log S)$. ***(6 marks)***

(b) Find an expression for the probability *in a risk-neutral world* of the event $a \leq S_T \leq b$. (express your answer in terms of the cumulative distribution function Φ of the standard normal distribution.) ***(8 marks)***

(c) Apply a risk-neutral valuation argument to find the spot value of this derivative. ***(3 marks)***

4 (i) Consider a market with risk-free return $r_B = 6\%$ and only two risky investments A and B. We are given the following data

Investment	**Expected return**	**Standard deviation of return**
A	25%	50%
B	10%	10%

and we are also told that the correlation between the returns of A and B is $\rho = 0$.

We assume that the CAPM holds.

(a) Use the fact that the market portfolio is the unique portfolio which maximizes

$$\frac{r_P - r_B}{\sigma_P}$$

as P ranges over all portfolios consisting entirely of risky investments to find the market portfolio in the market described above.

(7 marks)

(b) We are told that the total market value of A is £100,000,000. What is the total market value of B? ***(2 marks)***

(ii) Consider a market with risk-free return r_B and whose market portfolio M has expected return r_M and standard deviation of returns σ_M. Let A be an investment with expected return of r_A, standard deviation of returns σ_A and beta coefficient β.

(a) What is the slope of the capital market line? ***(2 marks)***

(b) Describe parametrically the curve c in the σ-r plane consisting of all points corresponding to investments spread between A and M.

(6 marks)

(c) Explain why c is tangent to the capital market line at the point M.

(3 marks)

(d) Use (c) to show that

$$r_A = \beta(r_M - r_B) + r_B.$$ ***(5 marks)***

End of Question Paper