The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2015–16**

Analytic Number Theory **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) Show that there are infinitely many primes of the form $3k-1$. ***(4 marks)***

(ii) Let $\Phi(x) = x^4 + x^3 + x^2 + x + 1$.

(a) Show that the set of primes which divide $\Phi(a)$ for some integer a is infinite. ***(3 marks)***

(b) Let $p > 5$ be a prime and assume that p divides $\Phi(a)$ for some integer a. Show that p divides $a^5 - 1$, and deduce that the residue class of a in $(\mathbb{Z}/p\mathbb{Z})^\star$ has order 5. ***(5 marks)***

(c) Deduce that that there are infinitely many primes of the forms $5n+1$. ***(3 marks)***

(iii) State Bertrand's Postulate. ***(1 mark)***

(a) Use Bertrand's Postulate to show that there are infinitely many primes beginning with the digits 1. ***(3 marks)***

(b) Use Bertrand's Postulate to show that

$$\frac{1}{m} + \frac{1}{m+1} + \ldots + \frac{1}{2m}$$

is never an integer for any integer $m \geq 1$. ***(6 marks)***

2 (i) Define the prime counting function and state the Prime Number Theorem. ***(1 mark)***

(a) Evaluate $\lim_{x\to\infty} \frac{\pi(ax)}{\pi(x)}$ where a is a positive real number. ***(4 marks)***

(b) Deduce that there is a constant C such that for any $x > C$ we can find at least 2015 distinct primes between x and $2x$. ***(5 marks)***

(ii) The Bernouilli numbers B_k are defined by the expression

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

(a) Prove that if $f(t) = \frac{t}{e^t - 1} + \frac{t}{2}$, then $f(t) = f(-t)$. ***(3 marks)***

(b) Deduce that $B_n = 0$ for odd $n \geq 3$. ***(3 marks)***

(iii) The Bernouilli polynomials $B_k(x)$ are defined by the expression

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}.$$

(a) Utilizing the power series expansion of $t/(e^t - 1)$ given in part (ii), show that $B_k(x) = \sum_{i=0}^{k} \binom{k}{i} B_i x^{k-i}$. ***(4 marks)***

(b) Show that $\frac{d}{dx}(B_k(x)) = kB_{k-1}(x)$ for all $k \geq 1$. ***(5 marks)***

3 (i) Let $f, g : \mathbb{N} \to \mathbb{C}$ be two arithmetic functions.

(a) Write down the formal Euler product expansion of $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ if f is multiplicative, and give a simplified version when f is completely multiplicative. ***(3 marks)***

(b) For an integer n, let $\sigma(n)$ be the sum of the positive divisors of n. Show formally that

$$\sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - \frac{1+p}{p^s} + \frac{p}{p^{2s}}}.$$

(6 marks)

(ii) (a) Define the Riemann zeta function $\zeta(s)$ and write down the Euler product for $\zeta(s)$, indicating in what region of the complex plane they are valid. ***(4 marks)***

(b) What is the Riemann Hypothesis? ***(2 marks)***

(iii) Recall the infinite product expansion

$$\frac{\sin z}{z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{(\pi n)^2}\right).$$

(a) By taking first the logarithm and then the derivative of both sides (ignoring convergence issues etc.), show that:

$$\frac{z \cos z}{\sin z} = 1 - 2 \sum_{k=1}^{\infty} \frac{\zeta(2k)}{\pi^{2k}} z^{2k}.$$

(4 marks)

(b) Using Euler's relation $e^{iz} = \cos z + i \sin z$, show that:

$$\frac{z \cos z}{\sin z} = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{B_{2k} 2^{2k}}{(2k)!} z^{2k}.$$

(4 marks)

Hence find an explicit closed form expression for $\zeta(2k)$ when $k \geq 1$. ***(2 marks)***

4 (i) Define a character of a finite abelian group. ***(1 mark)***

Show that if χ is a non-trivial character of a finite abelian group G, then

$$\sum_{g \in G} \chi(g) = 0.$$

(4 marks)

(ii) This question asks you to illustrate the proof of Dirichlet's Theorem in a specific case.

(a) List the characters of $(\mathbb{Z}/10\mathbb{Z})^\star$. ***(5 marks)***

(b) Prove that $L(1, \chi) \neq 0$ for each non-trivial character χ on your list. ***(7 marks)***

(c) Using the character table, show that for a prime p, we have

$$\sum_{\chi} \chi(\overline{7})^{-1} \chi(\overline{p}) = \begin{cases} 4 & \text{if } p \equiv 7 \mod 10 \\ 0 & \text{otherwise} \end{cases}$$

where the sum runs over the characters of $(\mathbb{Z}/10\mathbb{Z})^\star$. ***(2 marks)***

(d) Prove that there are infinitely many primes congruent to 7 modulo 10.

(You may assume that for any character χ, the sum $\displaystyle\sum_{p \neq 2,5} \sum_{n=2}^{\infty} \frac{\chi(p)}{np^{ns}}$ converges to a finite limit as $s \to 1$.) ***(6 marks)***

End of Question Paper