The University Of Sheffield.

**SCHOOL OF MATHEMATICS AND STATISTICS** **Autumn Semester 2015-2016**

**Complex Analysis** **2 hours 30 minutes**

*Answer* ***four*** *questions. If you answer more than four questions, only your best four will be counted.*

**1** (i) Express

$$\frac{(1+i\sqrt{3})^7}{(-1+i)^{17}}$$

in the form $re^{i\theta}$ with $r > 0$ and $-\pi < \theta \le \pi$. ***(4 marks)***

(ii) State, without proof, the triangle inequalities for $|\,z+w|$ and $|\,z-w|$. ***(1 mark)***

Let $D = \{z \in \mathbb{C} : |z-2i| < 1\}$. Prove that for all $z \in D$,

$$\sqrt{5}-1 \quad < \quad |z+1| \quad < \quad 1+\sqrt{5}.$$ ***(3 marks)***

Sketch $D$ and show that for all $z \in D$,

$$\left|\frac{\sin z}{z+1}\right| \quad < \quad \frac{\cosh 3}{\sqrt{5}-1}\,.$$ ***(6 marks)***

(iii) Find all the solutions of the following equation:

$$2\cos z + 4i\sin z = 1.$$ ***(6 marks)***

(iv) The path $\gamma$ consists of the straight line segment from $0$ to $2i$ followed by the straight line segment from $2i$ to $-2+2i$ followed by straight line segment from $-2+2i$ to $-2$. Evaluate

$$\int_\gamma (\mathrm{Re}\, z + \mathrm{Im}\, z + \cos z)\, dz\,.$$ ***(5 marks)***

**2** (i) Define what is meant by the following two statements:

(a) A function $f$ is **differentiable at the point** $z_0$ ;

(b) A function $f$ is **analytic in a region** $D$. ***(2 marks)***

Let

$$g(z) = \frac{z \sin z}{e^z (1 + \cos(\pi z))^3} \, .$$

Decide where $g$ is analytic giving reasons for your answer. ***(5 marks)***

(ii) State, without proof, the Cauchy-Riemann equations for a differentiable function. ***(1 mark)***

Let $h(z) = h(x+iy) = u(x,y)+iv(x,y)$ , where $u$ and $v$ are real valued functions. Prove that, if $h$ is analytic in $\mathbb{C}$ and $u$ and $v$ satisfy the relation $ve^u = 5$ everywhere then $h$ is constant. ***(6 marks)***

(iii) In each of the following cases, determine whether there is a function $k$ analytic on $\mathbb{C}$ with $\mathrm{Re}\,(k(x+iy)) = u(x,y)$, giving reasons for your answers:

$$\begin{aligned} &\text{(c)} \quad u(x,y) &= &\ x^2 + \sin x + \cosh y \, , \\ &\text{(d)} \quad u(x,y) &= &\ 3x^3 - 9xy^2 + 3x + 2 \, . \end{aligned}$$

When $k$ exists, find an explicit expression for $k(z)$ in terms of $z$ and show that you have found all the functions satisfying the conditions. ***(7 marks)***

(iv) The path $\alpha$ consists of the straight line segment from $-1$ to $-1+2i$ followed by the straight line segment from $-1+2i$ to $1+2i$ followed by the straight line segment from $1+2i$ to 1. Show that

$$\left| \int_\alpha \cos \bar{z} \, dz \right| \quad \leq \quad 6 \cosh 2.$$

***(4 marks)***

**3** State, without proof, Cauchy's Theorem and Cauchy's Integral Formulae for a function and for its derivatives. Your statement should include conditions under which the results are valid. ***(7 marks)***

Let $\gamma$ be the triangular contour with vertices $0,\ 2-2i,\ 2+2i$ described in the anti-clockwise direction. Without using the Residue Theorem, evaluate

(i) $\displaystyle\int_\gamma \frac{z^2 \sinh z}{z-1}\, dz$ , (ii) $\displaystyle\int_\gamma \frac{\cosh z}{4z^2-1}\, dz$ ,

(iii) $\displaystyle\int_\gamma \frac{(z^5+1)\cosh z}{z+1}\, dz$ , (iv) $\displaystyle\int_\gamma \frac{\cosh z}{(3z-1)^3}\, dz$ ,

(v) $\displaystyle\int_\gamma \mathrm{Re} z\, dz$ .

***(16 marks)***

Deduce that

$$\int_\gamma (\overline{z} + \cos z)\, dz = 8i.$$

***(2 marks)***

**4** (i) Let

$$f(z) = \frac{1}{(z-3)} \ .$$

Find the Taylor expansion of $f(z)$ about the point $-1$, giving an expression for the general term. ***(3 marks)***

Where is this expansion valid? ***(1 mark)***

(ii) Explain how Laurent expansions are used to classify isolated singularities. ***(5 marks)***

For each of the following functions, find **all the singularities** in $\mathbb{C}$. Classify these singularities giving reasons for your answers and evaluate the residue at each of them:

(a) $(z^2-1)\cos\left(\dfrac{1}{z-1}\right)$, ***(5 marks)***

(b) $\dfrac{1+e^{\pi i z}}{z-1}$ , ***(3 marks)***

(c) $\dfrac{\cos z + 2\cosh z}{z^7}$ , ***(4 marks)***

(d) $\dfrac{1+e^{\pi i z}}{(z+1)^3}$ . ***(4 marks)***

**5** (i) Let $\gamma$ be the triangular contour with vertices $4i, -4i, 2$ described in the anti-clockwise direction. Evaluate

$$\int_\gamma \frac{\cosh z}{1 + e^{i\pi z}}\, dz\ , \qquad \int_\gamma \frac{1}{(z-1)^2 \cos \pi z}\, dz\ ,$$ ***(14 marks)***

using Cauchy's Residue Theorem.

(ii) Let $\alpha > 0$. Prove that

$$\int_{-\infty}^{\infty} \frac{x^3 \sin \alpha x}{x^4 + 4}\, dx \quad = \quad \pi e^{-\alpha} \cos \alpha.$$ ***(11 marks)***

## End of Question Paper