The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn 2015**

Advanced Calculus and Linear Algebra **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

Throughout the paper E denotes an identity matrix.

1 (i) Let $(x, y) = F(u, v)$ and $(s, t) = G(x, y)$ be C^1 maps for which $G(F(u, v))$ is defined. State the Chain Rule for $G \circ F$.

Suppose now that $(x, y) = F(u, v)$ has an inverse map $(u, v) = F^{-1}(x, y)$ which is also C^1. Use the Chain Rule to obtain a formula for the derivative matrix $D(F^{-1})(F(u, v))$ in terms of the derivative matrix of F.

(6 marks)

(ii) Define *parabolic coordinates* for $(x, y) \in \mathbb{R}^2$ by

$$x = \tfrac{1}{2}(u^2 - v^2), \quad y = uv,$$

and define

$$(x, y) = F(u, v) = (\tfrac{1}{2}(u^2 - v^2),\ uv).$$

(a) Find the derivative matrix $D(F)(u, v)$ of F and the Jacobian $\dfrac{\partial(x, y)}{\partial(u, v)}$.

(b) Show that there is an inverse map $(u, v) = F^{-1}(x, y)$ defined for $y > 0$ and $u > 0$, $v > 0$. Find $D(F^{-1})(x, y)$ in terms of x, y.

(12 marks)

2 (i) Let $L\colon \mathbb{R}^p \to \mathbb{R}^q$ be a linear map.

(a) Define the kernel $\ker(L)$ and the image $\mathrm{im}(L)$ of L.

(b) Show that $\ker(L)$ is a vector subspace of $\mathbb{R}^p$, stating clearly the conditions for a subset of $\mathbb{R}^p$ to be a subspace.

(c) Define the rank and the nullity of L and state (but do not prove) the Rank-Nullity Theorem. **(10 marks)**

(ii) Let $L\colon \mathbb{R}^p \to \mathbb{R}$ and $L'\colon \mathbb{R}^p \to \mathbb{R}$ be linear maps. Write $V = \ker(L)$, and $V' = \ker(L')$.

(a) Define a linear map $L''\colon \mathbb{R}^p \to \mathbb{R}^2$ for which the kernel is $V \cap V'$. Prove that your map does have $V \cap V'$ as kernel.

(b) Now assume that L is not the zero map; that is, there is a nonzero vector $\mathbf{u} \in \mathbb{R}^p$ such that $L(\mathbf{u}) \neq 0$.

Show that the dimension d of $V \cap V'$ is either $p-2$ or $p-1$. **(9 marks)**

3 Let P denote the plane with equation $6x + 3y + 2z = 6$.

Using either a double or a triple integral, find the volume of the region bounded by P and the coordinate planes. **(6 marks)**

4 Cylindrical coordinates (r, θ, z) are defined for $(x, y, z) \in \mathbb{R}^3$ by

$$x = r\cos\theta, \quad y = r\sin\theta, \ z = z,$$

where $r \geqslant 0$ and $0 \leqslant \theta < 2\pi$.

A truncated cone is shown in Figure 1. The radius of the base is 2, the radius of the top is 1 and the height is 3. Using cylindrical coordinates, or otherwise, find the volume of the region inside the cone. **(7 marks)**

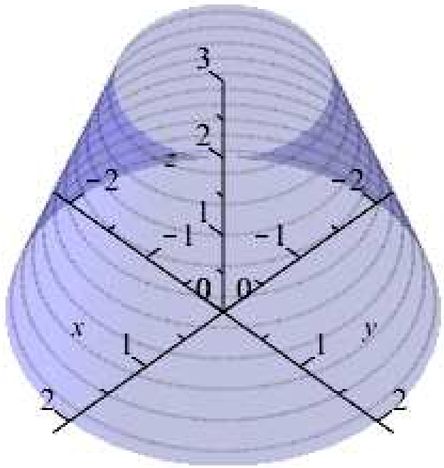

Figure 1: For Question 4

5 (a) Evaluate the integral

$$I_1 = \int_C \frac{1}{x^2 + y^2}\, \mathrm{d}s,$$

where s is a measure of arc length, for the following two paths C from point A $(1, 0)$ to point B $(0, 1)$:

(i) C is along the circumference of the circle centred on the origin and passing through A and B;

(ii) C is the straight line between A and B. **(8 marks)**

(b) State Green's Theorem, being careful to include any conditions needed for its validity. Use Green's Theorem to evaluate the line integral

$$I_2 = \oint_\Gamma \{-x^2 y \mathrm{d}x + y^2 x \mathrm{d}y\},$$

where Γ is the closed curve consisting of the semi-circle of radius a $\{(x, y) \mid x^2 + y^2 = a^2, y > 0\}$, and the segment $(-a, a)$ of the x-axis, described anti-clockwise. **(10 marks)**

 Question 5 continued on next page

5 (continued)

(c) Determine a function $f(y)$ for which the vector field $\mathbf{v}(x, y) = (f(y), x\cos y)$ is conservative, and find a corresponding potential function for $\mathbf{v}$. Evaluate the line integral

$$I_3 = \oint_\Gamma \mathbf{v} \cdot \mathrm{d}\mathbf{r},$$

when Γ is the triangular path with vertices $(0, 0)$, $(1, 0)$ and $(1, 1)$.
(7 marks)

6 (a) Find and classify all critical points of the function $f(x, y) = 2y^2x - x^2y + 4xy$.
(7 marks)

(b) Using a Lagrange multiplier, find the maximum distance from the origin $(0, 0)$ to the curve $3x^2 + 3y^2 + 4xy - 2 = 0$. **(8 marks)**

(c) Let $Q = 3x^2 + 3y^2 + 4xy$. Determine the symmetric 2×2 matrix A such that $Q = \mathbf{x}^\mathsf{T} A\mathbf{x}$, where $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$.
Find an orthogonal matrix P and a diagonal matrix D such that $A = PDP^\mathsf{T}$. Hence find a linear transformation

$$u = ax + by, \quad v = cx + dy,$$

such that $Q = u^2 + \alpha v^2$ for some constants a, b, c, d, and α which you should determine. Hence verify the result you obtained in part (b). **(10 marks)**

End of Question Paper