The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2015–2016**

Time Series **2 hours**

Marks will be awarded for your best ***three*** *answers.*

RESTRICTED OPEN BOOK EXAMINATION
Candidates may bring to the examination lecture notes and associated lecture material (but no textbooks) plus a calculator that conforms to University regulations.
There are 60 marks available on the paper.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

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1 (i) A model is to be fitted to a time series of length 81. Values of the sample autocorrelation function (ACF) and sample partial ACF (PACF) are tabulated below.

Lag (h)	1	2	3	4	5
ACF (r_h)	$\star$	0.7	0.05	0.02	0.01
PACF ($\hat{a}_h^{(h)}$)	0.9	$\star\star$	0.4	-0.15	0.10

(a) Find the omitted values ($\star$ and $\star\star$). ***(3 marks)***

(b) Check whether the time series is stationary. ***(1 mark)***

(c) Test whether the time series is consistent with white noise. ***(2 marks)***

(d) Test whether the time series is consistent with MA(1) and MA(2) moving average models. ***(5 marks)***

(e) Test whether the time series is consistent with AR(1), AR(2), AR(3) and AR(4) autoregressive models. ***(3 marks)***

(f) Giving your reason, state which of the models in (d) and (e) you prefer for these data. ***(2 marks)***

(ii) Consider the time series model

$$y_t = 2\sin\left(\frac{\pi}{t}\right) + \epsilon_t, \quad t = 1, 2, \ldots,$$

where ϵ_t follows a white noise process with variance 10.

(a) Show that y_t is non-stationary process. ***(2 marks)***

(b) Define an appropriate transformation of y_t to result in a stationary time series model. Justify your choice. ***(2 marks)***

2 Consider the time series model

$$y_t = c + \frac{1}{2}y_{t-1} + \epsilon_t - \frac{2}{3}\epsilon_{t-1},$$

where ϵ_t is white noise with variance 2.

(i) Write the above model using the backward shift operator B. ***(2 marks)***

(ii) Show that y_t is stationary. ***(2 marks)***

(iii) If $E(y_t) = 4$, then determine the constant c. ***(2 marks)***

(iv) Find the variance of y_t. ***(5 marks)***

(v) Given data $y_1, y_2, \ldots, y_n$ it is known that $\epsilon_{n-1} = 0.4$, $y_{n-1} = 4.2$ and $y_n = 4.5$.

(a) Show that

$$y_{n+2} = 3 + \frac{1}{4}y_n - \frac{1}{3}\epsilon_n - \frac{1}{6}\epsilon_{n+1} + \epsilon_{n+2}.$$

(2 marks)

(b) Based on the above data, calculate a 95% predictive interval for the observation y_{n+2}. ***(7 marks)***

3 Consider that y_t is generated by an ARMA(1,1) model

$$y_t = \alpha y_{t-1} + \epsilon_t + \beta\epsilon_{t-1},$$

where α, β are the AR and MA coefficients and ϵ_t is a Gaussian white noise with variance σ^2.

(i) Write down the likelihood and the log-likelihood functions of the parameters α, β and σ^2, based on a collection of observations $y_{1:n} = (y_1, y_2, \ldots, y_n)$. ***(6 marks)***

(ii) Using conditional least squares,

(a) derive the partial derivatives of the conditional log-likelihood with respect to α and β; ***(6 marks)***

(b) using part (a) show that the maximum likelihood estimates of α and β are

$$\hat{\alpha} = \frac{\sum_{t=2}^n y_t y_{t-1} \sum_{t=2}^n \epsilon_{t-1}^2 - \sum_{t=2}^n y_t\epsilon_{t-1} \sum_{t=2}^n y_{t-1}\epsilon_{t-1}}{\sum_{t=2}^n y_{t-1}^2 \sum_{t=2}^n \epsilon_{t-1}^2 - (\sum_{t=2}^n y_{t-1}\epsilon_{t-1})^2}$$

$$\hat{\beta} = \frac{\sum_{t=2}^n y_{t-1}^2 \sum_{t=2}^n y_t\epsilon_{t-1} - \sum_{t=2}^n y_{t-1}\epsilon_{t-1} \sum_{t=2}^n y_t y_{t-1}}{\sum_{t=2}^n y_{t-1}^2 \sum_{t=2}^n \epsilon_{t-1}^2 - (\sum_{t=2}^n y_{t-1}\epsilon_{t-1})^2}.$$

(8 marks)

4 Consider the ARMA(1,1) model for the time series y_t:

$$y_t = 0.2y_{t-1} + \epsilon_t + \epsilon_{t-1}, \qquad (1)$$

where ϵ_t follows Gaussian white noise with variance $\sigma^2 = 1$.

Define the state vector

$$\beta_t = \begin{bmatrix} y_t \\ \epsilon_{t+1} \\ \epsilon_t \end{bmatrix}.$$

Using this state vector write down model (1) in state space form, i.e.

$$y_t = x\beta_t + \eta_t$$
$$\beta_t = F\beta_{t-1} + \zeta_t$$

and determine x, F, η_t, ζ_t and the variances of η_t and ζ_t. ***(3 marks)***

(i) If the posterior distribution of the state β_1, given $y_1 = 2$ is

$$\beta_1 \mid \{y_1 = 2\} \sim N\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\},$$

then find the one-step forecast distribution of y_2. ***(6 marks)***

(ii) Using the result in (i) obtain a 95% predictive interval for y_2. ***(2 marks)***

(iii) At time $t = 2$ the observation y_2 is observed to be 3. Perform the Kalman filter iteration for $t = 2$ and obtain the posterior distribution of

$$\beta_2 \mid \{y_2 = 3\}.$$

(9 marks)

End of Question Paper