The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2015–2016**

Applicable Analysis **2 hours 30 minutes**

*Answer **four** questions. If you answer more than four questions, only your best four will be counted.*

You may use the following results when answering questions on this paper.

Function	*Laplace Transform*
$t^{\alpha}e^{bt} \quad (\alpha > -1)$	$\dfrac{\Gamma(\alpha+1)}{(s-b)^{\alpha+1}}$
$\sin at$	$\dfrac{a}{s^2+a^2}$
$\cos at$	$\dfrac{s}{s^2+a^2}$
$f(t)e^{bt}$	$F(s-b)$
$f^{(n)}(t)$	$s^nF(s) - \sum_{k=1}^{n} f^{(k-1)}(0)s^{n-k}$
$tf(t)$	$-F'(s)$

Please leave this exam paper on your desk
Do not remove it from the hall

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to be completed by student

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1 (i) Define what is meant by the statement that the improper integral $\int_a^\infty f(x)\,dx$ exists. ***(2 marks)***

(a) Prove **from your definition** that $\int_0^\infty 4\sin x\cos x\,dx$ does not exist. ***(2 marks)***

(b) **Using your definition**, find the values of α for which

$$\int_0^\infty \frac{2x}{(1+x^2)^\alpha}\,dx \quad \text{exists.}$$ ***(5 marks)***

(ii) State, without proof, the Comparison Test for convergence and divergence of integrals of the form $\int_a^\infty f(x)\,dx$. Your statement should include conditions under which the results are valid. ***(4 marks)***

Determine whether each of the following integrals converges or diverges, giving reasons for your answers.

(c) $$\int_0^\infty \frac{2x\cos x}{(x^2+1)^4}\,dx\,,$$

(d) $$\int_1^\infty \frac{1}{(x^3+7)^{\frac{1}{3}}}\,dx\,.$$

(7 marks)

(iii) (e) Determine whether

$$\int_0^1 \frac{e^x\cos x}{(1+x^2)\sqrt{x}}\,dx$$

converges or diverges, giving reasons for your answer.

(f) Determine whether

$$\int_0^1 \frac{e^x\cos x}{x\sqrt{x}}\,dx$$

converges or diverges, giving reasons for your answer. ***(5 marks)***

2 (i) State, without proof, the theorem concerning change of order in a repeated integral of the form

$$\int_c^d dy \int_a^\infty f(x,y)\,dx\,.$$

Your statement should include conditions under which the result holds.

(2 marks)

Let $0 < c < d$. Prove that

$$\int_c^d dy \int_0^\infty \frac{2y}{4x^2+y^2}\,dx \quad = \quad \int_0^\infty dx \int_c^d \frac{2y}{4x^2+y^2}\,dy\,.$$ ***(6 marks)***

Deduce that

$$\int_0^\infty \ln\left(\frac{4x^2+d^2}{4x^2+c^2}\right)\,dx \quad = \quad \frac{\pi(d-c)}{2}\,.$$ ***(7 marks)***

(ii) Define the Γ function. ***(2 marks)***

Prove that

(a) $$\int_1^\infty \frac{(\ln x)^5}{x\sqrt{x}}\,dx = 2^6(5!)$$ ***(4 marks)***

and

(b) $$\int_0^\infty x^9 e^{-4x^4}\,dx = \frac{3\sqrt{\pi}}{2^9}\,.$$ ***(4 marks)***

3 Define the Beta function. State, without proof, the relation between the Beta and Gamma functions. *(3 marks)*

Prove that

$$B(x,y) \quad = \quad 2\int_0^{\pi/2} \cos^{2x-1}\theta \sin^{2y-1}\theta \, d\theta \qquad (x>0,\, y>0)$$

and

$$B(x,y) \quad = \quad \int_0^{\infty} \frac{u^{x-1}}{(1+u)^{x+y}}\, du \qquad (x>0,\, y>0).$$

(4 marks)

Prove each of the following, stating any standard results you need to use:

(a) $$\int_0^{\pi/2} \frac{1}{\sqrt{\tan\theta}}\, d\theta \quad = \quad \frac{\pi}{\sqrt{2}}\ ;$$

(b) $$\int_0^{\infty} \frac{x\sqrt{x}}{(1+x^3)^2}\, dx \quad = \quad \frac{\pi}{9}\ ;$$

(c) $$\int_{-\infty}^{\infty} \frac{e^{2x}}{(e^{3x}+1)^2} \quad = \quad \frac{2\pi}{9\sqrt{3}}\ .$$ *(18 marks)*

4 (i) In each of the following cases, find the function continuous on $[0, \infty)$ with the given Laplace transform:

(a) $\dfrac{s-1}{s(s+1)} \quad (s > 0)\,;$

(b) $\dfrac{4s+2}{s^2+4} \quad (s > 0)\,.$

(5 marks)

(ii) Express

$$\frac{2(s+5)(s^2+4)}{s^2(s^2+9)}$$

in partial fractions. ***(5 marks)***

Suppose the functions f and g are continuous on $[0, \infty)$. Define the convolution $f*g$ and state, without proof, a relation between $L(f*g)$, $L(f)$ and $L(g)$. ***(3 marks)***

Using Laplace transforms, find the function f continuous on $[0, \infty)$ such that

$$f'(t) \; + \; 5\int_0^t f(u)\,\cos 2(t-u)\,du \quad = \quad 10 \quad (t \geq 0),$$

and f satisfies the initial condition $f(0) \; = \; 2\,.$ ***(12 marks)***

5 (i) Let $b > 0$. Using Beta and Gamma functions show that

$$\int_0^\infty \frac{x^3}{x^6 + b^2}\, dx = \frac{\pi}{3\sqrt{3}\, b^{2/3}} .$$ ***(7 marks)***

Prove that

$$\int_0^\infty \sin(x^3)\, dx = \frac{\pi}{3\sqrt{3}\,\Gamma(2/3)} .$$ ***(5 marks)***

(ii) Using Laplace Transforms solve the differential equation

$$t\frac{d^2y}{dt^2} + 3t\frac{dy}{dt} + (2t+1)y = e^{-2t}$$

subject to the conditions $y(0) = 1, \quad y(1) = \dfrac{2}{e^2}.$ ***(13 marks)***

End of Question Paper