The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring semester 2015-2016**

Operations Research **2 Hours**

Attempt all FOUR questions.

1 (i) A boat manufacturer produces two types of boats: a family rowboat and a sports canoe. The boats are molded from aluminum by means of a large pressing machine, and are finished by hand labor. A rowboat requires 50 kg of aluminum, 6 minutes of machine time, and 3 hours of finishing labor. A canoe requires 30 kg of aluminum, 5 minutes of machine time, and 5 hours of finishing labor. The company has available 1000 kg of aluminum, 300 minutes of machine time, and 200 hours of labor. The company gets £50 profit on the sale of a rowboat and £60 profit on a canoe. How many of each type should be manufactured in order to maximize total profits? Formulate the linear programming problem. **Find the formulation only; do NOT try to solve it**. ***(5 marks)***

(ii) Use the two-phase method to find the optimal solution for the following linear programming problem:

$$\max \quad z = -4x_2$$

subject to $x_1, x_2 \geq 0$ and

$$\begin{aligned} x_1 + x_2 &\leq 6, \\ 3x_1 + x_2 &\geq 3, \\ 2x_1 - 2x_2 &\leq 1. \end{aligned}$$

Clearly state your final solution. Hint: not counting the preprocessing step, you need three tableaux in phase 1. ***(20 marks)***

2 (i) A project manager is considering 10 investment opportunities and wishes to select the combination that maximises the profit. Let P_j denote the profit (in millions of £) for opportunity j ($j = 1, 2, ..., 10$). The requirements are:

- The manager has to make at least 6 investments.
- Investment opportunities 3 and 4 cannot be taken at the same time, neither can opportunities 5 and 6.
- Neither 5 nor 6 can be undertaken unless either 3 or 4 is undertaken.

Formulate the mixed integer linear programming (MILP) problem for the manager. **Find the formulation only; do NOT try to solve it**.

(10 marks)

(ii) A manufacturer is using two ingredients and a filler to produce a mixture for a dietary supplement tablet. The mixture may contain three types of nutrients: A, B and C. Several requirements have to be met:

- The mixture has to meet minimum requirements for *at least two types* of the nutrients. In one kilogram of the mixture, the minimum requirement is 75 grams for A, 50 grams for B, and 20 grams for C. If two nutrients have met the requirements, there is no requirement on the third one.
- The filler, though it contributes to the weight of the mixture, does not have nutritional contents, and its cost can be neglected.
- The manufacturer will need to produce 100 kilograms of the mixture.

The nutrient contents (unit: gram per kilogram of the ingredient) and the costs (unit: pound per kilogram of the ingredient) of each ingredient are given in the following table:

	A (g/kg)	B (g/kg)	C (g/kg)	Cost (£/kg)
Ingredient 1	100	80	40	4
Ingredient 2	75	150	20	6

Formulate the mixed integer linear programming problem with which the manufacturer can solve for the optimal amounts of the two ingredient in **1 kilogram** of the mixture. **Find the formulation only; do NOT try to solve it**.

(15 marks)

3 We consider a minimisation linear programming problem

$$\min\ z = c^T x, \quad \text{subject to} \quad Ax \geq b \quad \text{and} \quad x \geq 0.$$

(i) Define the Lagrangian function of the above problem and show that the dual linear programming problem is

$$\max\ v = b^T y, \quad \text{subject to} \quad A^T y \leq c \quad \text{and} \quad y \geq 0.$$

(14 marks)

(ii) Let $y^* = B^{-T} c_B$, where B is the optimal basic matrix for the primal problem, and c_B is the corresponding cost coefficient vector. Show that

$$A^T y^* \leq c.$$

(7 marks)

(iii) Write down the **two** sets of complementary slackness conditions corresponding to this pair of primal-dual problems. ***(4 marks)***

4 Cartoy is a manufacturer that produces two types of toy cars. The following information is known:

- Car 1 is sold at £25. It requires one hour's work from worker 1 and two hours' work from worker 2, and needs raw material worth £5.
- Car 2 is sold at £22. It requires two hours' work from worker 1 and one hour's work from worker 2, and needs raw material worth £4.
- Worker 1 can work up to 40 hours per week and is paid £5 per hour. Worker 2 can work up to 50 hours per week and is paid £6 per hour. As a consequence, the profit of producing a Car 1 is £3, and a Car 2 is £2.
- There is unlimited supply for raw material.

Let x_1 and x_2 be the numbers of car 1 and car 2 being produced, respectively, and x_3 and x_4 be the slack variables relating to the maximum working hours of worker 1 and worker 2, respectively. One can formulate the linear programming problem as follows:

$$\max \quad z = 3x_1 + 2x_2$$

subject to

$$\begin{aligned} x_1 + 2x_2 &\le 40 \\ 2x_1 + x_2 &\le 50 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Solving the problem with the simplex method, we find the following optimal tableau:

	x_1	x_2	x_3	x_4	Solution
z	0	0	1/3	4/3	80
x_1	1	0	−1/3	2/3	20
x_2	0	1	2/3	−1/3	10

(i) Using the information given in the optimal tableau, find the optimal solutions for the decision variables and the cost function, and the optimal solutions for the dual variables. ***(3 marks)***

(ii) Determine the range of the prices for car 1 for which the current basis remains optimal. ***(8 marks)***

(iii) If worker 2 can work only up to 15 hours per week due to medical conditions, would the current basis remain feasible and optimal? If not, find the new optimal feasible solution using the dual simplex method. Clearly state your new optimal solution. ***(16 marks)***

End of Question Paper