



Answer **four** questions. If you answer more than four questions, only your best four will be counted.

Throughout this paper, unless otherwise stated, all normed vector spaces are either over the field of real numbers, \mathbb{R} , or the field of complex numbers, \mathbb{C} .

1 (i) State what is meant by the statement that a normed vector space is a *Banach space*. (3 marks)

(ii) Let V be the space \mathbb{R}^k with the norm

$$\|(\alpha_1, \dots, \alpha_k)\| = (\alpha_1^4 + \dots + \alpha_k^4)^{\frac{1}{4}}.$$

Show that V is a Banach space. (6 marks)

(iii) Define what is meant by a *bounded linear operator* $T: V \rightarrow W$ between normed vector spaces V and W . (2 marks)

(iv) Show that a linear map $T: V \rightarrow W$, where V and W are normed vector spaces, is continuous if and only if it is a bounded linear operator. (9 marks)

(v) Let V be the vector space \mathbb{R}^k with the norm

$$\|(\alpha_1, \dots, \alpha_k)\| = |\alpha_1| + \dots + |\alpha_k|.$$

Show that every linear map $T: V \rightarrow W$ is continuous. (5 marks)

2 Let H be a complex Hilbert space. Throughout this question, you may use without proof the fact that if V is a closed linear subspace of H , then $H = V \oplus V^\perp$. You may also use without proof the Cauchy-Schwarz inequality, and the fact that the inner product is continuous.

(i) For a linear subspace $U \subseteq H$, prove that $\overline{U}^\perp = U^\perp$, and U^\perp is closed. (6 marks)

(ii) Let $f: H \rightarrow \mathbb{C}$ be a continuous linear map. Prove that we have a unique element $R_f \in H$ such that

$$f(x) = \langle R_f, x \rangle$$

for all $x \in H$. (9 marks)

(iii) Let l^2 be the Hilbert space

$$\left\{ (a_n) \mid \sum_{n=1}^{\infty} |a_n|^2 < \infty \right\}$$

with the usual inner product. Define $f, g: l^2 \rightarrow \mathbb{C}$ by

$$f((a_n)) = a_1 \quad g((a_n)) = \sum_{n=4}^{\infty} \frac{a_n}{n}.$$

Find R_f , R_g , and R_{f+ig} . (5 marks)

(iv) Let $T: H \rightarrow H$ be a bounded linear map, and let $U \subseteq H$ be a closed linear subspace. Is it necessarily true that $T[U^\perp] = T[U]^\perp$? Justify your answer. (5 marks)

3 (i) State Zorn's lemma. (4 marks)

(ii) Define what is meant by an *orthonormal basis* for a Hilbert space H . Prove that every Hilbert space has an orthonormal basis. (11 marks)

(iii) Define the *graph* of a linear map $T: V \rightarrow W$. State the closed graph theorem. (4 marks)

(iv) Let H be a Hilbert space, and let $f, g: H \rightarrow H$ be linear maps such that

$$\langle f(x), y \rangle = \langle x, g(y) \rangle$$

for all $x, y \in H$. Prove that f and g are both continuous. (6 marks)

4 (i) Let A be a unital Banach algebra, and let $x \in A$. Prove tht if $\|x\| < 1$, then the element $1 - x$ is invertible. (8 marks)

(ii) Define the *spectrum* of an element $x \in A$, and the *spectral radius* $R_\sigma(x)$. (3 marks)

(iii) Prove that for any $x \in A$, $R_\sigma(x) \leq \|x\|$. (4 marks)

(iv) Let H be a Hilbert space. Say what is meant by the statement that a bounded linear map $U: H \rightarrow H$ is a *unitary operator*. (2 marks)

(v) Prove that if $U: H \rightarrow H$ is a unitary operator, then $\|U\| = 1$. (4 marks)

(vi) Prove that if $U: H \rightarrow H$ is unitary, then

$$\text{Spectrum}(U) \subseteq \{z \in \mathbb{C} \mid |z| = 1\}.$$

(4 marks)

5 (i) Let V be a normed vector space. Define what is meant by a *compact operator* $K: V \rightarrow V$. (3 marks)

(ii) Prove that any bounded linear map with finite-dimensional image is a compact operator. You may use the Heine-Borel theorem without proof. (4 marks)

(iii) Prove that any compact operator is a bounded linear map. (4 marks)

(iv) Give an example of a bounded linear map that is *not* a compact operator. Justify your answer. (4 marks)

(v) Let H be a Hilbert space. Define what is meant by a *Fredholm operator* $T: H \rightarrow H$ and its *Fredholm index*. (3 marks)

(vi) Prove that each of the following operators on the Hilbert space l^2 are Fredholm, and calculate their Fredholm indices. In the calculation, you may use any standard properties of the Fredholm index you need.

(a) $R(a_1, a_2, a_3, a_4, a_5, \dots) = (0, 0, a_1, a_2, a_3, a_4, a_5, \dots)$

(b) $S(a_1, a_2, a_3, a_4, a_5, a_6, a_7, \dots) = (a_1, a_2, a_1, a_2, a_3, a_4, a_5, a_6, a_7, \dots)$

(c) $T(a_1, a_2, a_3, a_4, a_5, \dots) = (0, 0, 0, a_2, a_3, a_4, a_5, \dots)$

(7 marks)

End of Question Paper