The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring semester 2014-2015**

Advanced Operations Research **2 Hours**

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

1 (i) Use the two-phase method to find the optimal solution for the following linear programming problem:

$$\max \quad z = -2x_1 - 3x_2$$

subject to $x_1, x_2 \geq 0$ and

$$\begin{aligned} x_1 + x_2 &\leq 4, \\ 3x_1 + 2x_2 &\geq 6, \\ x_1 - x_2 &\leq 1. \end{aligned}$$

Clearly state your final solution. Hint: not counting the preprocessing step, you need three tableaux in phase 1. ***(15 marks)***

(ii) Use the dual simplex method to find the optimal solution for the same problem above. ***(10 marks)***

2 (i) A company produces products A and B by processing material M through a machine. The requirements and selling price of a unit of each product are given as follows

	M(units)	Machine time (min)	Selling Price (£)
A	5	2	25
B	8	6	45

The company has available 120 minutes of machine time weekly at no cost. The material M, however, must be purchased from an outside vendor. The company can purchase no more than 350 units of M per week. The price is £3/unit for the first 200 units, and £2/unit afterward.

Defining x_1 and x_2 to be the number of units of A and B, respectively, to be produced and sold weekly, x_3 the number of units of M purchased at £3/unit and x_4 the number purchased at £2/unit, formulate the mixed integer-linear programming problem to maximise the revenue of the company. You may introduce additional binary variables where necessary. Do NOT try to solve it, but explain briefly why the formulation is correct.

(15 marks)

(ii) A company uses three factories to produce four types of office desks. The following table lists the different types of desks that can be produced by each factory.

Factory	Desk types
1	1,2,3
2	2,3
3	1,3,4

All desks require same per-unit labor and material. The daily capacities of the three factories are 250, 180 and 300 desks, respectively. The daily demands for the four types of desks are 200, 150, 350 and 100 units, respectively. The company wants to use a maximal flow model to find the production schedule that will most satisfy the total demand of the desks. Sketch and annotate properly the network of the problem. Do NOT try to find the mathematical formulation, nor the solution. ***(10 marks)***

3 (i) Given a maximisation linear programming problem $\max z = c^T x$, subject to $Ax \geq b$, $x \geq 0$, show that its dual linear programming problem is

$$\min v = -y^T b, \quad \text{subject to} \quad -A^T y \geq c, y \geq 0.$$

(14 marks)

(ii) Let $y^* = -B^{-T} c_B$, where B is the optimal basic matrix for the primal problem, and c_B is the corresponding cost coefficient vector. Show that

$$-A^T y^* \geq c.$$

(11 marks)

4 (i) The payoff matrix for a two-person zero-sum game is given as follows:

$$A = \begin{bmatrix} -1 & 3 & 2 & 0 \\ 4 & -4 & -3 & 5 \\ -2 & 2 & -1 & 1 \end{bmatrix},$$

where the rows represent the pure strategies for player A and the columns represent those for player B.

(a) Show that the game has no pure strategy equilibrium solution. ***(2 marks)***

(b) Use dominance to simplify the payoff matrix, then use the graphical method to find the optimal strategies for the players. ***(17 marks)***

(ii) Let $A = (a_{ij})$ $(1 \le i \le m,\ 1 \le j \le n)$ be the payoff matrix of a general $m \times n$ two-person zero-sum game, and $X = (x_1, x_2, ..., x_m)$ be the mixed strategy of the row player A. Given that the security level for X is

$$g(X) = \min_{1 \le j \le n} \sum_{i=1}^{m} x_i a_{ij},$$

derive the linear programming problem from which you can solve for the optimal strategy for player A. Do NOT try to solve the linear programming problem. ***(6 marks)***

5 A workshop uses mitre saw and hammer drill to produce three types of wooden furniture F1, F2, and F3. The table below summarises the pertinent data:

Tools	Time for F1 (m/unit)	Time for F2 (m/unit)	Time for F3 (m/unit)	Capacity(m)
Saw	2	5	3	5300
Drill	3	4	6	5400
Unit price(£)	3	6	5	

in which time is measured in minutes (m). To determine the production schedule that maximises the total revenue, we define x_1, x_2 and x_3 as the numbers of units of F1, F2, and F3 to be produced, respectively, and formulate the following linear programming model:

$$\max \quad z = 3x_1 + 6x_2 + 5x_3$$

subject to $x_1, x_2, x_3 \geq 0$, and

$$2x_1 + 5x_2 + 3x_3 \leq 5300,$$
$$3x_1 + 4x_2 + 6x_3 \leq 5400.$$

Introducing slack variable x_4 for the first constraint and x_5 for the second constraint, the optimal tableau is found as follows:

Basis	x_1	x_2	x_3	x_4	x_5	Solution
z	0	0	1/7	6/7	3/7	48000/7
x_2	0	1	-3/7	3/7	-2/7	5100/7
x_1	1	0	18/7	-4/7	5/7	5800/7

(i) From the optimal tableau, find the optimal cost, optimal solution for the primal variables, and the optimal solution for the dual variables. ***(4 marks)***

(ii) Which constraints are binding? Why? ***(2 marks)***

(iii) Suppose the capacity of the drill can be increased at a cost of 0.5£/minute. Is it profitable to increase the available drill time? ***(3 marks)***

(iv) How much do we have to increase the price of F3 before it is profitable to produce at the optimal solution? ***(4 marks)***

(v) Suppose that the time needed to use the drill to produce a unit of F2 may be changed to $(4 + \delta)$ m/unit from the current value 4 m/unit, and at the same time, the time needed to use the drill to produce a unit of F1 may be changed to $(3 + 2\delta/5)$ m/unit. Find the range of values for δ for which the optimal basis remains the same. ***(12 marks)***

End of Question Paper