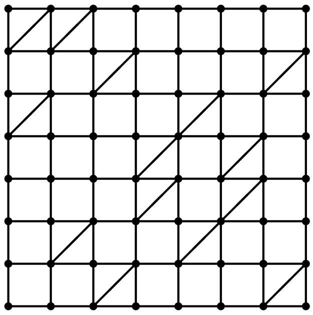
The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2014–2015**

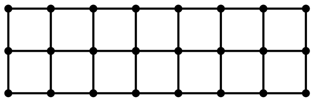
Graph Theory **2 hours 30 minutes**

Answer ***four*** *questions. You are advised* ***not*** *to answer more than four questions: if you do, only your best four will be counted.*

1 (i) (a) By drawing a suitable graph, determine whether the braced framework shown below is rigid. If it is rigid, is it a minimum bracing? If it is not rigid, what is the minimum number of additional braces needed to make it rigid? Justify your answers. ***(6 marks)***

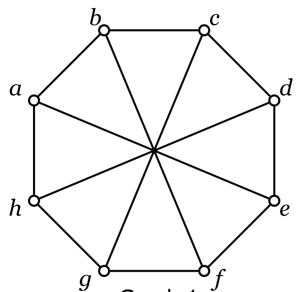


(b) What is the minimum number of braces needed for a rigid bracing of the framework shown below? How many ways are there to create a rigid bracing with this number of braces? Justify your answers. ***(7 marks)***

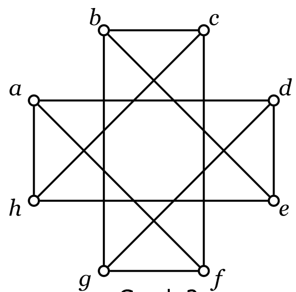


(ii) A compound has chemical formula $C_6H_{17}X_3$, where X is an unknown element. If the molecule is a tree, calculate the valency of X. ***(4 marks)***

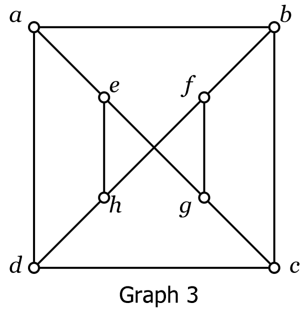
(iii) Define an *isomorphism* of graphs. Which of the following graphs are isomorphic? Justify your answer. ***(8 marks)***



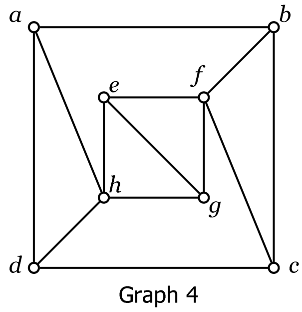
Graph 1



Graph 2



Graph 3



Graph 4

2 (i) The distances between seven towns are given in the table below.

A						
70	B					
62	57	C				
66	47	58	D			
78	66	59	95	E		
80	83	61	99	79	F	
75	64	71	65	40	42	G

(a) Use the nearest-insertion heuristic algorithm, starting at A, to find a good upper bound on the travelling salesman problem for these towns. You should make it clear which edge you are adding at each stage. ***(4 marks)***

(b) By omitting A, give a good lower bound for the travelling salesman problem. ***(4 marks)***

(ii) A project consists of ten tasks. The duration in days of each task and the other tasks which must precede it are given in the table.

Task	Duration	Preceding tasks
A	12	C, F
B	5	C, G
C	12	—
D	3	J, K
E	10	A, B
F	5	G
G	7	—
H	3	F
I	3	A
J	4	B, I
K	2	H, I

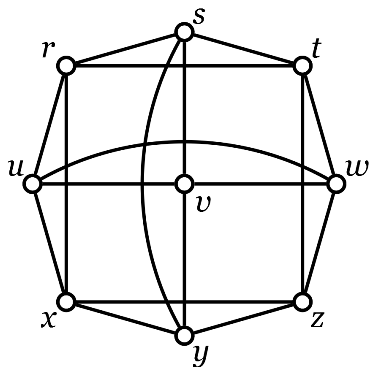
(a) Use Fulkerson's algorithm to construct an activity network. Find the shortest possible time for completion of the project, the earliest start time for each task, and the latest time each task can be started if the project is to finish in the shortest possible time. ***(10 marks)***

(b) Is there a task such that the project could be completed faster if the duration for that task was reduced? If so, which tasks have that property? Justify your answers. ***(4 marks)***

(c) Suppose there are not enough workers available to work on more than two tasks simultaneously. Explain why the project can no longer be completed in the minimum time found above. ***(3 marks)***

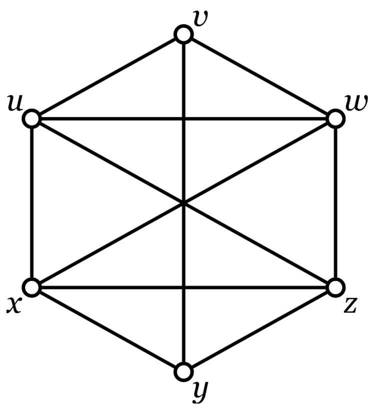
3 (i) State a necessary and sufficient condition for there to exist a partition of the edges of a graph G into disjoint cycles. Show that if G is a plane graph with this property, then it is possible to colour the faces of G with two colours such that faces meeting along an edge are different colours. ***(5 marks)***

(ii) (a) State Kuratowski's Theorem. Show that the graph H below is not planar by finding a suitable subgraph. Show how it may be drawn on a torus without edges crossing. Your drawing should use the labelling of the vertices given. ***(9 marks)***



(b) How many faces does your drawing of H have? How many colours are required to colour the faces such that faces meeting along an edge are different colours? Give a colouring using the minimum number of colours. ***(4 marks)***

(iii) Show how the graph F below may be drawn on a Möbius strip without edges crossing (you should label the vertices in your drawing). How many colours are required to colour the faces of your drawing such that faces meeting along an edge are different colours? Give a colouring using the minimum number of colours. ***(7 marks)***



4 (i) Suppose G is a simple connected plane graph with n vertices, all of degree 4, such that every vertex is surrounded by one face of degree a and three of degree b, for some constants $a < b$.

(a) How many faces of degree a and degree b must G have? ***(3 marks)***

(b) State Euler's formula for a connected plane graph, and use it to show that $\dfrac{1}{a} + \dfrac{3}{b} > 1$. ***(5 marks)***

(c) Hence deduce the values of a and b. How many vertices, edges and faces does G have? ***(8 marks)***

(d) Give an example of a connected but non-simple graph with all vertices having degree 4, such that every vertex is surrounded by one face of degree c and three faces of degree d, for some constants $c < d$. ***(2 marks)***

(ii) Seven ice hockey teams, A–G, are required to play thirteen matches as given in the table below (a cross in the table indicates that those teams must play each other).

A	×	×	–	–	–	×
	B	×	×	–	×	–
		C	×	×	–	–
			D	×	×	–
				E	×	×
					F	×
						G

The matches are to be scheduled so that no team plays more than one match in any week. Relate this problem to a graph, and state the parameter of the graph which gives the minimum number of weeks needed. Determine (with justification) the minimum number of weeks, and give an example of a schedule which achieves this. ***(7 marks)***

5 (i) Let G be a simple graph, and x and y be non-adjacent vertices. Explain what is meant by $G_{x=y}$. State and prove a relationship between the chromatic polynomials of G, $G_{x=y}$ and $G + xy$. ***(5 marks)***

(ii) Write down the chromatic polynomial of C_3 (in factorised form), and use the relationship in (i) to find the chromatic polynomial of C_4. ***(5 marks)***

(iii) For $n > 4$, use the relationship in (i) to show that

$$P_{C_n}(k) = (k-a)P_{C_{n-1}}(k) + (k-b)P_{C_{n-2}}(k)\,,$$

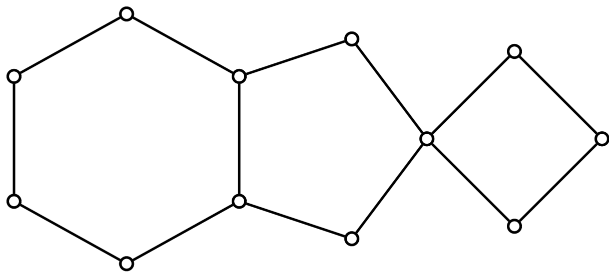
where a and b are constants you should determine. ***(5 marks)***

(iv) Hence prove, by induction or otherwise, that

$$P_{C_n}(k) = (k-1)^n + (-1)^n(k-1)$$

for every $n \geq 3$. ***(7 marks)***

(v) Find the chromatic polynomial of the graph shown (it is not necessary to simplify the polynomial). ***(3 marks)***



End of Question Paper