The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2014–2015**

Mathematical Methods **2 hours**

Marks will be awarded for your best ***FOUR*** *answers. The marks awarded to each question or section of question are shown in italics.*

1 (a) A function $f(x)$ is defined for $-\infty < x < \infty$ by

$$f(x) = e^{-|x|}.$$

Show that the Fourier transform, $\hat{f}(k)$, of $f(x)$ for real k is given by

$$\hat{f}(k) = \frac{2}{k^2+1}.$$ ***(6 marks)***

(b) A function $g(x)$ is defined for $-\infty < x < \infty$ by

$$g(x) = \sin x.$$

Find the Fourier transform, $\hat{g}(k)$, of $g(x)$ for real k. ***(4 marks)***

[You may assume that

$$\int_{-\infty}^{\infty} e^{ikx}\,dx = 2\pi\delta(k).$$]

(c) By using the inverse Fourier transform

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\,\hat{f}(k)\,dk,$$

or otherwise, show that

$$\mathcal{F}\{f(x)g(x)\} = \frac{1}{2\pi}\left(\hat{f} * \hat{g}\right)(k),$$

where $\mathcal{F}$ denotes the Fourier transform, and the convolution $\left(\hat{f} * \hat{g}\right)(k)$ is defined by

$$\left(\hat{f} * \hat{g}\right)(k) = \int_{-\infty}^{\infty} \hat{f}(s)\hat{g}(k-s)\,ds.$$ ***(9 marks)***

(d) Using the results of parts (a), (b) and (c) show that

$$\mathcal{F}\left\{e^{-|x|}\sin x\right\} = i\left[\frac{1}{(k-1)^2+1} - \frac{1}{(k+1)^2+1}\right]$$

for real k. ***(6 marks)***

2 The Laplace transform of a function $f(t)$ is defined by

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

(a) By using the change of variables $t^{1/2} = u$, show that for $\operatorname{Re} s > 0$

$$\mathcal{L}\left\{t^{-1/2}\right\} = \sqrt{\pi}\, s^{-1/2}.$$ ***(3 marks)***

$$\left[\ \text{You may assume that} \quad \int_0^\infty e^{-su^2}\, du = \frac{1}{2}\sqrt{\frac{\pi}{s}} \quad \text{for } \operatorname{Re} s > 0.\ \right]$$

Hence, by integrating by parts, show that, if n is a positive integer and $\operatorname{Re} s > 0$,

$$\mathcal{L}\left\{t^{n-1/2}\right\} = \sqrt{\pi}\,\frac{1}{2}\cdot\frac{3}{2}\cdots\left(n-\frac{3}{2}\right)\left(n-\frac{1}{2}\right) s^{-\left(n+\frac{1}{2}\right)}.$$ ***(7 marks)***

(b) Find the Laplace transform of $\cos\omega t$ for $\operatorname{Re} s > 0$. ***(4 marks)***

(c) If $F(t)$ is defined for $t > 0$ by

$$F(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau,$$

show that

$$\mathcal{L}\{F(t)\} = \mathcal{L}\{f(t)\}\,\mathcal{L}\{g(t)\}.$$ ***(5 marks)***

(d) Use the results of parts (a), (b) and (c) to show that the inverse Laplace transform of

$$\frac{s^{-5/2}}{s^2+1}$$

is

$$\frac{8}{15\sqrt{\pi}} \int_0^t \tau^{5/2} \cos(t-\tau)\, d\tau.$$ ***(6 marks)***

3 The function $y(x)$ satisfies the ordinary differential equation

$$x^2y'' - 2xy' + 2y = x^3e^{-x} \qquad (1)$$

in $-1 < x < 2$, with $y(-1) = y(2) = 0$, where $y' = \dfrac{dy}{dx}$ etc.

(a) By trying $y = x^n$, find the independent solutions of

$$x^2y'' - 2xy' + 2y = 0.$$

(4 marks)

(b) Given that Green's function $G(x;\xi)$ for the boundary-value problem given at the beginning of the question is continuous at $x = \xi$, and that $\partial G/\partial x$ has a discontinuity of size $1/\xi^2$ at $x = \xi$, show that

$$G(x;\xi) = \begin{cases} \dfrac{\xi-2}{3\xi^3}(x^2+x) & -1 \leqslant x < \xi, \\[2ex] \dfrac{\xi+1}{3\xi^3}(x^2-2x) & \xi < x \leqslant 2. \end{cases}$$

(15 marks)

(c) Using Green's function, show that the solution to equation (1) which satisfies the boundary conditions given at the beginning of the question is

$$y(x) = xe^{-x} - \frac{1}{3}(2e+e^{-2})x + \frac{1}{3}(e-e^{-2})x^2.$$

(6 marks)

[You may use

$$\int_{-1}^{x}(\xi+1)e^{-\xi}\,d\xi = e - (2+x)e^{-x} \quad \text{and} \quad \int_{x}^{2}(\xi-2)e^{-\xi}\,d\xi = (x-1)e^{-x} - e^{-2}.$$]

4 Consider the equation

$$(1+\epsilon)x^2 - 2x - 3 = 0, \qquad (*)$$

where ϵ is a constant satisfying $0 < \epsilon \ll 1$.

(a) The solution to equation $(*)$ can be written as

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots,$$

where $x_0, x_1, x_2, \ldots$ are $O(1)$ as $\epsilon \to 0$.

Use this expression to derive the two solutions to equation $(*)$, correct to order ϵ^2 as $\epsilon \to 0$. ***(18 marks)***

(b) Find the exact solutions of $(*)$, and show that their expansions agree with your results from part (a). ***(7 marks)***

5 The complementary error function, $\mathrm{erfc}(x)$, is defined for $x > 0$ by

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt.$$

By changing variables, show that

$$\mathrm{erfc}(x) = \frac{e^{-x^2}}{\sqrt{\pi}} \int_0^\infty e^{-xv}\, e^{-v^2/4}\, dv.$$ ***(3 marks)***

Expand $e^{-v^2/4}$ in powers of v and change variables by $u^2 = xv$. Hence, defining

$$I_n = \int_0^\infty u^{2n+1}\, e^{-u^2}\, du,$$

show that

$$\sqrt{\pi}\, x\, e^{x^2}\, \mathrm{erfc}(x) \sim 2 \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(4x^2)^k}\, I_{2k} \qquad \text{as } x \to \infty.$$ ***(7 marks)***

Show that $I_n = nI_{n-1}$ for $n > 0$, and hence find an asymptotic expansion for $\mathrm{erfc}(x)$ as $x \to \infty$. ***(10 marks)***

For the second term in the asymptotic expansion to make less than a 1% relative change to the first term, show that $x > \sqrt{50}$. ***(5 marks)***

End of Question Paper