The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring semester 2014-2015**

Operations Research **2 Hours**

Answer ***four*** *questions. You are advised* ***not*** *to answer more than four questions: if you do, only your best four will be counted.*

1 Use the two-phase method to find the optimal solution for the following linear programming problem:

$$\max \quad z = -3x_1 - 4x_2$$

subject to $x_1, x_2 \geq 0$ and

$$x_1 + x_2 \leq 5,$$
$$3x_1 + x_2 \geq 3,$$
$$x_1 + x_2 \geq 2,$$
$$x_1 + 4x_2 \geq 4.$$

Hint: you will need four tableaux in phase 1 and 1 tableau in phase 2, excluding the preprocessing steps. ***(25 marks)***

2 (i) A company produces products A and B by processing material M through a machine. The requirements and selling price of a unit of each are given as follows

	M(units)	Machine time (min)	Selling Price (£)
A	5	2	25
B	8	6	45

The company has available 120 min of machine time weekly at no cost. The material M, however, must be purchased from an outside vendor. The company can purchase no more than 350 units of M per week. The price is £3/unit for the first 200 units, and £2/unit afterward.

Define x_1 and x_2 to be the number of units of A and B, respectively, to be produced and sold weekly, x_3 the number of units of M purchased at £3/unit and x_4 the number purchased at £2/unit. The objective is to maximize the revenue

$$z = 25x_1 + 45x_2 - 3x_3 - 2x_4.$$

Introducing additional binary variables where necessary, formulate the constraints to obtain a mixed integer-linear programming problem. Do NOT try to solve it, but explain briefly why the formulation is correct.

(19 marks)

(ii) Write down the complementary slackness conditions for the following pair of primal and dual linear programming problems:

$$\begin{aligned}&\text{Max } z(x) = c^T x, \quad Ax \le b, \quad x \ge 0,\\ &\text{Min } v(y) = b^T y, \quad A^T y \ge c, \quad y \ge 0.\end{aligned}$$

Use the conditions to show that the dual variables are zero for non-binding constraints, and that, if a variable is non-zero at the optimal solution, then its reduced cost must be zero. ***(6 marks)***

3 (i) Starting from the definition of the Lagrangian function, derive the dual problem for the following linear programming problem:

$$\max \quad z = -2x_1 - 3x_2$$

subject to $x_1, x_2 \ge 0$ and

$$\begin{aligned}x_1 + x_2 &\le 4,\\ 3x_1 + 2x_2 &\ge 6,\\ x_1 - x_2 &\le 1.\end{aligned}$$

(13 marks)

(ii) Use the dual simplex method to find the optimal solution of the above primal linear programming problem. ***(12 marks)***

4 The payoff matrix for a two-person zero-sum game is given as follows:

$$A = \begin{bmatrix} -1 & 3 & 2 & 0 \\ 4 & -4 & -3 & 5 \\ -2 & 2 & -1 & 1 \end{bmatrix},$$

where the rows represent the pure strategies for player A and the columns represent those for player B.

(i) Show that the game has no pure strategy equilibrium solution.

(4 marks)

(ii) Use dominance to simplify the payoff matrix, then use the graphical method to find the optimal strategies for the players. ***(21 marks)***

5 A workshop uses mitre saw and hammer drill to produce three types of wooden furnitures F1, F2, and F3. The table below summarises the pertinent data:

Tools	Time for F1 (m/unit)	Time for F2 (m/unit)	Time for F3 (m/unit)	Capacity(m)
Saw	2	5	3	5300
Drill	3	4	6	5400
Unit price(£)	3	6	5	

in which time is measured in minutes (m). To determine the production schedule that maximises the total revenue, we define x_1, x_2 and x_3 as the numbers of units of F1, F2, and F3 to be produced, respectively, and formulate the following linear programming model:

$$\max \quad z = 3x_1 + 6x_2 + 5x_3$$

subject to $x_1, x_2, x_3 \geq 0$, and

$$2x_1 + 5x_2 + 3x_3 \leq 5300,$$
$$3x_1 + 4x_2 + 6x_3 \leq 5400.$$

Introducing slack variable x_4 for the first constraint and x_5 for the second constraint, the optimal tableau is found as follows:

Basis	x_1	x_2	x_3	x_4	x_5	Solution
z	0	0	1/7	6/7	3/7	48000/7
x_2	0	1	-3/7	3/7	-2/7	5100/7
x_1	1	0	18/7	-4/7	5/7	5800/7

(i) From the optimal tableau, find the optimal cost, optimal solution for the primal variables, and the optimal solution for the dual variables. ***(4 marks)***

(ii) Which constraints are binding? Why? ***(2 marks)***

(iii) Suppose the capacity of the drill can be increased at a cost of 0.5£/minute. Is it profitable to increase the available drill time? ***(3 marks)***

(iv) How much do we have to increase the price of F3 before it is profitable to produce at the optimal solution? ***(4 marks)***

(v) Suppose that the time needed to use the drill to produce a unit of F2 may be changed to $(4 + \delta)$ m/unit from the current value 4 m/unit, and at the same time, the time needed to use the drill to produce a unit of F1 may be changed to $(3 + 2\delta/5)$ m/unit. Find the range of values for δ for which the optimal basis remains the same. ***(12 marks)***

End of Question Paper