The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2014–2015**

Fluid Mechanics I **2 hours**

Answer all four questions.

1 (i) Consider a Newtonian fluid with the following stress tensor σ_{ij}

$$\sigma_{ij} = C_{ij} + C_{ijkl}e_{kl},$$

where C_{ij}, C_{ijkl} are constants and e_{kl} the strain rate tensor.
Assuming isotropy of fluid, we can write

$$C_{ij} = A\delta_{ij},$$

$$C_{ijkl} = B\delta_{ij}\delta_{kl} + D\delta_{ik}\delta_{jl} + E\delta_{il}\delta_{jk},$$

where A, B, D, E are constants. Derive

$$\sigma_{ij} = (-p + \lambda e_{kk})\delta_{ij} + 2\mu e_{ij}$$

by suitably defining three constants p, λ and μ. Find the trace σ_{ii} of σ_{ij}. By defining $\zeta = \lambda + \frac{2}{3}\mu$, eliminate λ from σ_{ij}. ***(8 marks)***

(ii) Write down the stress tensor for an incompressible fluid with constant density ρ using the results obtained in part (i). ***(2 marks)***

(iii) Assuming that the characteristic length and velocity scales of an incompressible viscous fluid with kinematic viscosity ν are L and U, respectively, derive the expression for Reynolds number Re and explain briefly its physical meaning.

(5 marks)

(iv) Prove the following identity

$$\nabla^2 \mathbf{v} = \nabla(\nabla \cdot \mathbf{v}) - \nabla \times \boldsymbol{\omega}$$

where $\boldsymbol{\omega} = \nabla \times \mathbf{v}$. ***(5 marks)***

(v) For a boundary layer of thickness δ, the velocity profile u in the boundary layer is given by

$$\frac{u}{U_0} = \left(\frac{y}{\delta}\right)^{1/2}$$

where U_0 is a constant. The displacement thickness δ_1 can be expressed as

$$\delta_1 = \int_0^\delta \left(1 - \frac{u}{U_0}\right) dy$$

and the momentum thickness δ_2 as

$$\delta_2 = \int_0^\delta \frac{u}{U_0}\left(1 - \frac{u}{U_0}\right) dy.$$

Determine the ratios δ_1/δ and δ_2/δ.

(5 marks)

2 Consider steady flow of incompressible fluid of density ρ flowing along an infinitely long fixed horizontal pipe of circular cross-section of radius a, and let Oz be along the axis. Take axes $Oxyz$ with Oz in the direction of the flow and assume that gravity acts in the $-x$ direction. Further assume that the only non-zero component of velocity is the z-component, w, i.e. $\mathbf{v} = (0, 0, w)$.

(i) Show that

$$\frac{\partial w}{\partial z} = 0.$$

(1 mark)

(ii) Show that the Navier-Stokes equations reduce to

$$0 = -g - \frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{1}$$

$$0 = \frac{\partial p}{\partial y}, \tag{2}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu \nabla^2 w. \tag{3}$$

(5 marks)

(iii) Using equations (1) and (2), show that

$$p = -\rho g x + f(z),$$

where $f(z)$ is an arbitrary function. ***(2 marks)***

(iv) By assuming the value of the constant pressure gradient in z-direction (obtained in part (iii)) to be $-P_0$ ($P_0 > 0$), calculate the velocity profile from equation (3). (You can use $\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$ for this part.)

(6 marks)

(v) Calculate the drag per unit area on the wall of the tube. ***(4 marks)***

(vi) Compute rate of flow (or volume flux) along the tube. ***(3 marks)***

(vii) Explain how the result obtained in part (vi) can be used to confirm the no-slip boundary conditions on the pipe wall. ***(4 marks)***

3 You are given the two-dimensional Navier-Stokes equations and the continuity equation in terms of cylindrical polar co-ordinates (r, θ):

$$\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu \left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} \right),$$

$$\frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu \left(\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} \right),$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0,$$

where

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Consider a steady flow in the region between two infinite coaxial circular cylinders of radii a, b $(a < b)$, where the inner cylinder is rotating with constant angular velocity Ω_1 and the outer cylinder is rotating with constant angular velocity Ω_2. We assume that the velocity field is given by

$$\mathbf{u} = u(r)\mathbf{e}_\theta.$$

(i) Confirm that the continuity equation is satisfied. ***(2 marks)***

(ii) Simplify each component of the Navier-Stokes equations, assuming that $\dfrac{\partial p}{\partial \theta} = 0$. ***(6 marks)***

(iii) Show that the equation for $u(r)$ can be written as

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(ru) \right) = 0. \qquad (*)$$

(4 marks)

(iv) By solving the above equation $(*)$ show that the velocity has the form

$$u(r) = \frac{1}{b^2 - a^2}\left(\frac{a^2 b^2 (\Omega_1 - \Omega_2)}{r} + (b^2\Omega_2 - a^2\Omega_1) r \right). \quad (a \le r \le b).$$

(8 marks)

(v) Hence, calculate the pressure $p(r)$ in $a \le r \le b$. ***(5 marks)***

4 (i) The relative velocity between a fluid and a solid boundary is zero. A *boundary layer* is a thin layer of fluid adjacent to the boundary where the fluid velocity changes so that the free flow velocity is attached at its outer edge. Viscous and inertial forces are of comparable magnitudes within the boundary layer. Consider a flow with free stream velocity u_0 past a flat horizontal plate and assume

(a) unsteady flow,
(b) two-dimensional flow,
(c) no body forces,
(d) Reynolds number $Re \gg 1$,
(e) constant density ρ,
(f) pressure and inertial forces are comparable.

From the Navier-Stokes equations for an incompressible fluid, derive the boundary layer equations for the velocity $\mathbf{u} = (u, v)$ and pressure p.

(10 marks)

(ii) A solid sphere of radius a and centre O is moving with constant velocity $\mathbf{V}$ in a viscous incompressible fluid, and is not rotating. The velocity $\mathbf{u}$ and pressure p of the fluid outside the sphere are given by:

$$\mathbf{u} = \alpha \left(\frac{3a}{4r} + \frac{a^3}{4r^3} \right) \mathbf{V} + \left(\frac{3a}{4r^3} - \frac{3a^3}{4r^5} \right) (\mathbf{V} \cdot \mathbf{x})\mathbf{x} \qquad (**)$$

where α is a constant.

(a) By using the no-slip boundary condition on $r = a$, show that $\alpha = 1$.

(2 marks)

(b) Show that $\mathbf{u}$ in equation $(**)$ above satisfies the continuity equation.

(8 marks)

(c) You are given that the stress vector $\mathbf{t}$ on $r = a$ is given by

$$t_i = -p_0 n_i - \frac{3\mu}{2a} V_i,$$

where p_0 is the pressure far from the sphere. Calculate the total force exerted on the sphere by the fluid. ***(5 marks)***

End of Question Paper